

Mathematics
Higher level
Paper 3 – discrete mathematics

Friday 18 November 2016 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 17]

In this question the notation $(a_n a_{n-1} \dots a_2 a_1 a_0)_b$ is used to represent a number in base b , that has unit digit of a_0 . For example $(2234)_5$ represents $2 \times 5^3 + 2 \times 5^2 + 3 \times 5 + 4 = 319$ and it has a unit digit of 4.

- (a) Let x be the cube root of the base 7 number $(503231)_7$.
- (i) By converting the base 7 number to base 10, find the value of x , in base 10.
- (ii) Express x as a base 5 number. [7]
- (b) Let y be the base 9 number $(a_n a_{n-1} \dots a_1 a_0)_9$. Show that y is exactly divisible by 8 if and only if the sum of its digits, $\sum_{i=0}^n a_i$, is also exactly divisible by 8. [7]
- (c) Using the method from part (b), find the unit digit when the base 9 number $(321321321)_9$ is written as a base 8 number. [3]

2. [Maximum mark: 8]

In this question no graphs are required to be drawn. Use the handshaking lemma and other results about graphs to explain why,

- (a) a graph cannot exist with a degree sequence of 1, 2, 3, 4, 5, 6, 7, 8, 9; [2]
- (b) a simple, connected, planar graph cannot exist with a degree sequence of 4, 4, 4, 4, 5, 5; [3]
- (c) a tree cannot exist with a degree sequence of 1, 1, 2, 2, 3, 3. [3]

3. [Maximum mark: 16]

In a computer game, Fibi, a magic dragon, is climbing a very large staircase. The steps are labelled $0, 1, 2, 3 \dots$

She starts on step 0. If Fibi is on a particular step then she can either jump up one step or fly up two steps. Let u_n represent the number of different ways that Fibi can get to step n . When counting the number of different ways, the order of Fibi's moves matters, for example jump, fly, jump is considered different to jump, jump, fly. Let $u_0 = 1$.

(a) Find the values of u_1, u_2, u_3 . [3]

(b) Show that $u_{n+2} = u_{n+1} + u_n$. [2]

(c) (i) Write down the auxiliary equation for this recurrence relation.

(ii) Hence find the solution to this recurrence relation, giving your answer in the form $u_n = A\alpha^n + B\beta^n$ where α and β are to be determined exactly in surd form and $\alpha > \beta$. The constants A and B do not have to be found at this stage. [5]

(d) (i) Given that $A = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)$, use the value of u_0 to determine B .

(ii) Hence find the explicit formula for u_n . [3]

(e) Find the value of u_{20} . [1]

(f) Find the smallest value of n for which $u_n > 100\,000$. [2]

Turn over

4. [Maximum mark: 19]

The simple, complete graph $\kappa_n (n > 2)$ has vertices $A_1, A_2, A_3, \dots, A_n$. The weight of the edge from A_i to A_j is given by the number $i + j$.

- (a) (i) Draw the graph κ_4 including the weights of all the edges.
- (ii) Use the nearest-neighbour algorithm, starting at vertex A_1 , to find a Hamiltonian cycle.
- (iii) Hence, find an upper bound to the travelling salesman problem for this weighted graph. [4]
- (b) Consider the graph κ_5 . Use the deleted vertex algorithm, with A_5 as the deleted vertex, to find a lower bound to the travelling salesman problem for this weighted graph. [5]

Consider the general graph κ_n .

- (c) (i) Use the nearest-neighbour algorithm, starting at vertex A_1 , to find a Hamiltonian cycle.
- (ii) Hence find and simplify an expression in n , for an upper bound to the travelling salesman problem for this weighted graph. [7]
- (d) By splitting the weight of the edge $A_i A_j$ into two parts or otherwise, show that all Hamiltonian cycles of κ_n have the same total weight, equal to the answer found in (c)(ii). [3]