Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2016

## Discrete mathematics

## Higher level

## Paper 3

10 pages



International Baccalaureate®
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## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a valid **Method**; working must be seen.

***(M)*** Marks awarded for **Method**; may be implied by **correct** subsequent working.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***(A)*** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

***R*** Marks awarded for clear **Reasoning**.

***N*** Marks awarded for **correct** answers if **no** working shown.

***AG*** Answer given in the question and so no marks are awarded.

### Using the markscheme

**1 General**

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2016**". It is essential that you read this document before you start marking.
In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

| | **Correct answer seen** | **Further working seen** | **Action** |
|---|---|---|---|
| **1.** | $8\sqrt{2}$ | $5.65685\ldots$ *(incorrect decimal value)* | Award the final ***A1*** *(ignore the further working)* |
| **2.** | $\frac{1}{4}\sin 4x$ | $\sin x$ | Do not award the final ***A1*** |
| **3.** | $\log a - \log b$ | $\log(a-b)$ | Do not award the final ***A1*** |

## 3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

## 4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

## 5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

**6 Misread**

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

**7 Discretionary marks *(d)***

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

**8 Alternative methods**

*Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.*

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

**9 Alternative forms**

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  **Example**: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textit{\textbf{A1}}$$

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

**10 Accuracy of Answers**

*Candidates should* ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for* ***FT***.

**11 Crossed out work**

*If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.*

**12 Calculators**

*A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.*

**Calculator notation** The mathematics HL guide says:
*Students must always use correct mathematical notation, not calculator notation.*
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

**13 More than one solution**

*Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.*

**1.** (a) (i) converting to base 10

$(503231)_7 = 5\times7^5 + 3\times7^3 + 2\times7^2 + 3\times7 + 1 = 85184$ ***M1A1A1***

so $x = 44$ ***A1***

(ii) repeated division by 5 gives ***(M1)***

| 44 | remainder |
|---|---|
| 8 | 4 |
| 1 | 3 |
| | 1 |

***(A1)***

so base 5 value for $x$ is $(134)_5$ ***A1***

**Notes:** Alternative method is to successively subtract the largest multiple of 25 and then 5.
Follow through if they forget to take the cube root and obtain $(10211214)_5$ then award ***(M1)(A1)A1***.

***[7 marks]***

(b) $9 \equiv 1 \pmod 8$ ***A1***

$9^i \equiv 1^i \equiv 1 \pmod 8 \qquad i \in \mathbb{N}$ ***(M1)(A1)***

$y = a_n 9^n + a_{n-1}9^{n-1} + \ldots + a_1 9 + a_0 \equiv a_n 1^n + a_{n-1}1^{n-1} + \ldots + a_1 1 + a_0 \equiv$

$a_n + a_{n-1} + \ldots + a_1 + a_0 \equiv \sum_{i=0}^{n} a_i \pmod 8$ ***M1A1A1***

so $y \equiv 0 \pmod 8$ and hence divisible by 8 if and only if $\sum_{i=0}^{n} a_i \equiv 0 \pmod 8$

and hence divisible by 8 ***R1AG***

***[7 marks]***

**Note:** Accept alternative valid methods *eg* binomial expansion of $(8+1)^i$, factorization of $(a^i - 1)$ if they have sufficient explanation.

(c) using part (b), $(321321321)_9 \equiv 3+2+1+3+2+1+3+2+1 = 18 \equiv 2 \pmod 8$

***M1A1***

so the unit digit is 2 ***A1***

***[3 marks]***

***Total [17 marks]***

**2.** (a) assume such a graph exists
by the handshaking lemma the sum of the degrees equals twice the number of edges ***R1***
but $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ which is odd ***R1***
this is a contradiction so graph does not exist ***AG***
***[2 marks]***

(b) assume such a graph exists
since the graph is simple and connected (and $v = 6 > 2$) then $e \le 3v - 6$ ***R1***
by the handshaking lemma $4 + 4 + 4 + 4 + 5 + 5 = 2e$ so $e = 13$ ***A1***
hence $13 \le 3 \times 6 - 6 = 12$ ***A1***
this is a contradiction so graph does not exist ***AG***
***[3 marks]***

(c) assume such a graph exists
a tree with 6 vertices must have 5 edges (since V-E+1=2 for a tree) ***R1***
using the Handshaking Lemma $1 + 1 + 2 + 2 + 3 + 3 = 2 \times 5$ implies
$12 = 10$ ***M1A1***
this is a contradiction so graph does not exist ***AG***
***[3 marks]***

***Total [8 marks]***

**3.** (a) $u_1 = 1,\ u_2 = 2,\ u_3 = 3$ ***A1A1A1***
***[3 marks]***

(b) to get to step $n + 2$ she can either fly from step $n$ or jump from step $n + 1$ ***R2***
so $u_{n+2} = u_{n+1} + u_n$ ***AG***
***[2 marks]***

(c) (i) auxiliary equation $\lambda^2 - \lambda - 1 = 0$ ***M1A1***

| **Note:** Award ***M1*** for attempting to write down a relevant quadratic. |
|---|

(ii) $\lambda = \dfrac{1 \pm \sqrt{1 + 4}}{2}$ ***M1A1***

$u_n = A\left(\dfrac{1 + \sqrt{5}}{2}\right)^n + B\left(\dfrac{1 - \sqrt{5}}{2}\right)^n$ ***A1***

***[5 marks]***

*continued…*

*Question 3 continued*

(d) (i) $u_0 = 1$ implies $A + B = 1$. So $B = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$ ***M1A1***

(ii) $u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ ***A1***

**Note:** Accept equivalent expressions in parts (i) and (ii).

***[3 marks]***

(e) $u_{20} = 10946$ ***A1***

***[1 mark]***

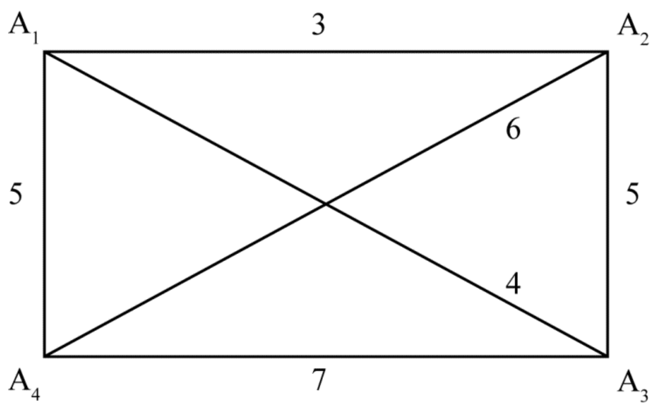
(f) using table, smallest value for *n* is 25 (gives 121393) ***(M1)A1***

**Note:** Accept other methods, *eg*, logs on the dominant term.

***[2 marks]***

***Total [16 marks]***

**4.** (a) (i)




***A1A1***

**Note:** ***A1*** for the graph, ***A1*** for the weights.

(ii) cycle is $A_1A_2A_3A_4A_1$ ***A1***

(iii) upper bound is $3 + 5 + 7 + 5 = 20$ ***A1***

***[4 marks]***

(b) with $A_5$ deleted, (applying Kruskal's Algorithm) the minimum spanning tree will consist of the edges $A_1A_2$, $A_1A_3$, $A_1A_4$, of weights 3, 4, 5 ***(M1)A1***

the two edges of smallest weight from $A_5$ are $A_5A_1$ and $A_5A_2$ of weights 6 and 7 ***(M1)A1***

so lower bound is $3 + 4 + 5 + 6 + 7 = 25$ ***A1***

***[5 marks]***

*continued…*

*Question 4 continued*

(c) (i) starting at $A_1$ we go $A_2$, $A_3$ … $A_n$
we now have to take $A_nA_1$
thus the cycle is $A_1A_2A_3 \ldots A_{n-1}A_nA_1$ ***A1A1***

| **Note:** Final ***A1*** is for $A_nA_1$. |
|---|

(ii) smallest edge from $A_1$ is $A_1A_2$ of weight $3$, smallest edge from $A_2$ (to a new vertex) is $A_2A_3$ of weight $5$, smallest edge from $A_{n-1}$ (to a new vertex) is $A_{n-1}A_n$ of weight $2n-1$ ***(M1)***
weight of $A_nA_1$ is $n+1$
weight is $3 + 5 + 7 + \ldots + (2n-1) + (n+1)$ ***A1***
$= \frac{(n-1)}{2}(2n+2) + (n+1)$ ***M1A1***
$= n(n+1)$ (which is an upper bound) ***A1***

| **Note:** Follow through is not applicable. |
|---|

***[7 marks]***

(d) put a marker on each edge $A_iA_j$ so that $i$ of the weight belongs to vertex $A_i$ and $j$ of the weight belongs to vertex $A_j$ ***M1***
the Hamiltonian cycle visits each vertex once and only once and for vertex $A_i$ there will be weight $i$ (belonging to vertex $A_i$) both going in and coming out ***R1***
so the total weight will be $\sum_{i=1}^{n} 2i = 2\frac{n}{2}(n+1) = n(n+1)$ ***A1AG***

| **Note:** Accept other methods for example induction. |
|---|

***[3 marks]***

***Total [19 marks]***