United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1

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Solutions

1. Find three even numbers less than 400, each of which can be expressed as a sum of consecutive positive odd numbers in at least six different ways.

(Two expressions are considered to be different if they contain different numbers. The order of the numbers forming a sum is irrelevant.)

SOLUTION

2. One day Arun and Disha played several games of table tennis. At five points during the day, Arun calculates the percentage of the games played so far that he has won. The results of these calculations are 30%, 40%, 50%, 60% and 70% in some order. What is the smallest possible number of games they played?

SOLUTION

REMARK

There are many possible constructions that work with 30 games:

- 3/5, 3/10, 12/20, 21/30 with 50% won after 2, 4, 14 or 16 games.
- 3/5, 3/10, 6/15, 21/30 with 50% won after 2, 4, 6 or 18 games.
- 2/5, 7/10, 9/15, 9/30 with 50% won after 2, 4, 6 or 18 games.
- 3/5, 7/10, 8/20, 9/30 with 50% won after 2, 4, 6, 14 or 16 games.

3. For each integer $0 \le n \le 11$, Eliza has exactly three identical pieces of gold that weigh 2^n grams. In how many different ways can she form a pile of gold weighing 2021 grams?

(*Two piles are different if they contain different numbers of gold pieces of some weight. The arrangement of the pieces in the piles is irrelevant.*)

SOLUTION

4. Two circles Γ_1 and Γ_2 have centres O_1 and O_2 respectively. They pass through each other's centres and intersect at A and B. The point C lies on the minor arc BO_2 of Γ_1. The points D and E lie on the line O_2C such that $\angle AO_1D = \angle DO_1C$ and $\angle CO_1E = \angle EO_1B$. Prove that triangle DO_1E is equilateral.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

SOLUTION

5. An N-set is a set of different positive integers including a given positive integer N. Let $m(N)$ be the smallest possible mean of any N-set. For how many values of N less than 2021 is $m(N)$ an integer?

SOLUTION

6. Marvin has been tasked with writing down every list of integers with the following properties:

(i) The list contains 72 terms.

(ii) The first term is 1.

(iii) Every term after the first is equal to either the previous term, or the sum of all previous terms.

When Marvin is finished, how many of the lists will have a sum divisible by 999,999?

SOLUTION

We begin by studying general lists satisfying conditions (ii) and (iii), and leave the condition on the length till later.

After the first term in the list, each subsequent term is either equal to the previous term, which we will denote by P, or the total of the preceding terms, which we will denote by T.

Moreover, since the second term will be 1 either way, we may assume, without loss of generality, that the list of letters which determines the list of numbers begins with a T.

For example the ten letters $TPPTTPPPTP$ give us the 11 numbers 1, 1, 1, 1, 1, 5, 10, 10, 10, 40, 40 which have total sum of 120.

We can view the list of letters as being composed of *blocks* starting with a single T and followed by some number (possibly zero) of Ps.

Now suppose we have a list numbers with sum Σ and we extend the corresponding list of letters by adding one of these blocks. If the block has length k (and thus contains $k - 1$ copies of P) the list of numbers will be extended by adding exactly k copies of the number Σ. Thus, the sum of all the numbers in the extended list will be $(k + 1)\Sigma$.

This means that if we have list of letters composed of blocks of lengths $j_1, j_2, ..., j_n$, then the sum of the corresponding list of numbers will be $(j_1 + 1)(j_2 + 1) \dots (j_n + 1)$. In our example we have blocks of length 3, 1, 4 and 2 giving a sum of $4 \times 2 \times 5 \times 3 = 120$ as expected.

Now we turn to the problem at hand. We have that $j_1 + j_2 + \cdots + j_n = 71$ and that $(j_1 + 1)(j_2 + 1) \dots (j_n + 1)$ is divisible by $999,999 = 3^3 \times 7 \times 11 \times 13 \times 37$.

One possibility is to have $2, 2, 2, 6, 10, 12$ and 36 among the j_i. These sum to 70, leaving room for one more block of length 1.