United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2
Wednesday 25 January 2023

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
4. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
5. Start each question on an official answer sheet on which there is a QR code.
6. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Do not write your name on any additional sheets.**
7. You should write in blue or black ink, but may use pencil and other colours for diagrams.
8. Write on one side of the paper only. Make sure your writing and diagrams are not too faint.
9. You may hand in rough work where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
10. Arrange your answer sheets, including rough work, in question order before they are collected. If you are not submitting work for a particular problem, remove the associated answer sheet.
11. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 27 January. Candidates in time zones more than 3 hours ahead of GMT must sit the paper on Thursday 26 January (as defined locally).
12. Around 24 high-scoring students eligible to represent the UK at the International Mathematical Olympiad, will be invited to a training session held in Cambridge around the Easter holidays.
13. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. Let ABC be a triangle with an obtuse angle A and incentre I. Circles ABI and ACI intersect BC again at X and Y respectively. The lines AX and BI meet at P, and the lines AY and CI meet at Q. Prove that $BCQP$ is cyclic.

2. For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible:

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1, 2, 3, 4 \xrightarrow{(i)} 2, 3, 1, 4 \xrightarrow{(ii)} 2, 3, 4, 1$

Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

3. For an integer $n \geq 3$, we say that $A = (a_1, a_2, \ldots, a_n)$ is an *n-list* if every a_k is an integer in the range $1 \leq a_k \leq n$. For each $k = 1, \ldots, n-1$, let M_k be the minimal possible non-zero value of $\left| \frac{a_1+\ldots+a_{k+1}}{k+1} - \frac{a_1+\ldots+a_k}{k} \right|$, across all n-lists. We say that an n-list A is *ideal* if

$$\left| \frac{a_1 + \ldots + a_{k+1}}{k+1} - \frac{a_1 + \ldots + a_k}{k} \right| = M_k$$

for each $k = 1, \ldots, n-1$.

Find the number of ideal n-lists.

4. The side lengths a, b, c of a triangle ABC are positive integers such that the highest common factor of a, b and c is 1. Given that $\angle A = 3\angle B$ prove that at least one of a, b and c is a cube.

