Cayley

1. Each of Alice and Beatrice has their birthday on the same day.

In 8 years' time, Alice will be twice as old as Beatrice. Ten years ago, the sum of their ages was 21.

How old is Alice now?

Solution

Let Alice be a years old now and let Beatrice be b years old now.

In 8 years' time, Alice will be twice as old as Beatrice, so that

$$a + 8 = 2(b + 8),$$

which we may simplify to

$$a - 2b = 8. \qquad (1)$$

Ten years ago, the sum of Alice's age and Beatrice's age was 21, so that

$$a - 10 + b - 10 = 21,$$

which we may rewrite in the form

$$a + b = 41. \qquad (2)$$

Adding equation (1) to 2 × equation (2) in order to eliminate b, we get $3a = 90$, so that $a = 30$.

Hence Alice is now 30 years old.

2. In the addition shown, each of the letters D, O, G, C, A and T represents a different digit.

What is the value of $D + O + G + C + A + T$?

$$\begin{array}{r} D\,O\,G \\ +\,C\,A\,T \\ \hline 1\,0\,0\,0 \end{array}$$

Solution

We first note that when we add two different single digits, the result is between $0 + 1$ and $8 + 9$, that is, the sum is between 1 and 17.

Starting with the 'ones' column of the addition in the question, G and T add to give a result that is between 1 and 17 and with 0 as its 'ones' digit, so their total is 10, that is,

$$G + T = 10. \qquad (1)$$

Moving to the 'tens' column, we have to include the 1 carried from the 'ones' column so that

$$O + A + 1 = 10,$$

that is,

$$O + A = 9 \qquad (2)$$

and therefore 1 is carried forward.

Similarly, in the 'hundreds' column, we include the 1 carried forward and have

$$D + C + 1 = 10$$

so that

$$D + C = 9. \qquad (3)$$

Summing the three equations (3) to (1), we obtain $D + O + G + C + A + T = 28$.

Is a solution possible?

In the diagram alongside, for example, $D + O + G + C + A + T = 28$, as required.

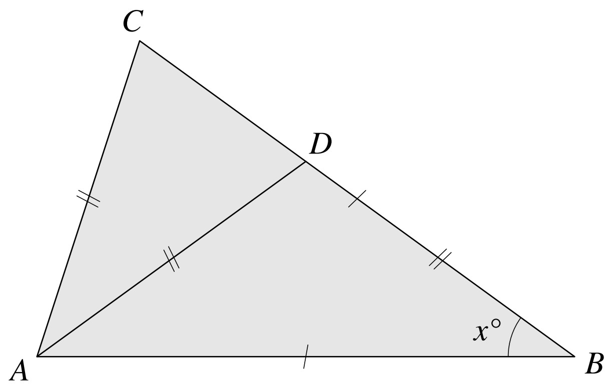
$$\begin{array}{r} 321 \\ +679 \\ \hline 1000 \end{array}$$

3. The triangle ABC is isosceles with $AB = BC$. The point D is a point on BC, between B and C, so that $AC = AD = BD$.

What is the size of angle ABC?

Solution

> Commentary
>
> For this solution, we use the results that
>
> (a) 'the angles opposite the two equal sides of an isosceles triangle are equal' (base angles of an isosceles triangle),
>
> (b) 'an exterior angle of a triangle is equal to the sum of the two opposite interior angles' (exterior angle of a triangle),
>
> (c) 'the angle sum of a triangle is 180° ' (angle sum of a triangle).



Let angle ABC be $x°$, as shown.

Triangle ABD is isosceles since $AD = BD$, so that $\angle ABD = \angle DAB = x°$ (base angles of an isosceles triangle).

Thus, from triangle ABD, we have $\angle ADC = 2x°$ (exterior angle of a triangle).

Triangle ACD is isosceles with $AC = AD$ so that $\angle ADC = \angle ACD = 2x°$ (base angles of an isosceles triangle).

Also, triangle ABC is isosceles with $AB = BC$ so that $\angle BCA = \angle CAB = 2x°$ (base angles of an isosceles triangle).

Now, considering triangle ABC, we have $2x + x + 2x = 180$ (angle sum of a triangle).

Hence $x = 36$ and so angle ABC is $36°$.

4. Arrange the digits 1, 2, 3, 4, 5, 6, 7, 8 to form two 4-digit integers whose difference is as small as possible.

Explain clearly why your arrangement achieves the smallest possible difference.

Solution

Let the two numbers be '$abcd$' and '$efgh$', with the smaller being '$abcd$'.

For a difference of less than 1000,

$$e = a + 1, \qquad (*)$$

so that the two 4-digit numbers are in consecutive thousands. All other choices have a difference of more than 1000.

For '$efgh$' to be as little above '$e000$' as possible, we use 1, 2 and 3 for f, g and h respectively.

Similarly, for'$abcd$' to be as close as possible to, but below, '$e000$' we use 8, 7 and 6 for b, c and d respectively.

This leaves 4 and 5, which satisfy (*), to be the values of a and e respectively.

Therefore the smallest difference comes from $5123 - 4876$, which is 247.

5. Howard chooses n different numbers from the list 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, so that no two of his choices add up to a square.

What is the largest possible value of n?

Solution

> Commentary
>
> When compiling the list of pairs which add up to a square, we can either work through the list (what numbers pair with 2?, what pairs with 3?, . . .), or note that we are looking for squares between 2 + 3 and 10 + 11, that is, 9 and 16.

When considering which numbers Howard should choose, we have to avoid the following pairs of numbers since their sum is a square: 2 and 7, 3 and 6, 4 and 5, 5 and 11, 6 and 10, and 7 and 9.

By considering, for example, the first three pairings, no two of which have a number in common, we see that there is no group of numbers with less than three members with the property that one of the numbers in the group is contained in each of the above six pairings.

However, each of the above six pairings contains one of the three numbers 5, 6 or 7.

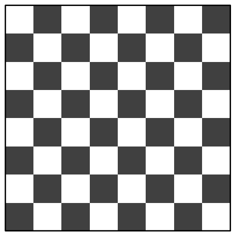
> Commentary
>
> Not choosing 5, 6 and 7 means that one of the numbers from all six pairings has been left out, so each pairing will not be included.
>
> So, for the largest possible list, Howard should choose 2, 3, 4, 8, 9, 10 and 11.

Thus the smallest number of individual numbers that need to be avoided so that none of these pairings is present is three.

Hence the largest possible value of n is seven.

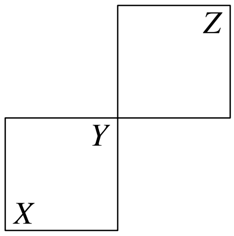
6. A chessboard is formed from an 8×8 grid of alternating black and white squares, as shown. The side of each small square is 1 cm.

What is the largest possible radius of a circle that can be drawn on the board in such a way that the circumference is entirely on white squares or corners?



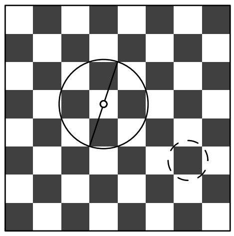
SOLUTION

To go from one white square to another, it is necessary to pass through a corner.



But it is not possible for a circle to pass through the point where two white squares meet (such as Y in the diagram alongside) and both the diagonally opposite corners of this point (X and Z), since these points form a straight line and not an arc of a circle.

Is it possible for a circle to pass through exactly one of X and Z, for *every* choice of Y on the circle? In other words, the circle is alternately "close" to a diagonal and an edge of white squares.



The diagram alongside shows one such circle, whose centre is as shown. The circle has diameter $\sqrt{1^2 + 3^2}$, from Pythagoras' theorem.

A circle passing through Y and neither X nor Z, for *every* point Y on the circle, such as the dashed circle in the diagram alongside, has a smaller radius.

Therefore the largest possible radius of the circle is $\frac{1}{2}\sqrt{10}$.