UKMT

United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Hamilton paper

Thursday 17 March 2022

England & Wales: Year 10
Scotland: S3
Northern Ireland: Year 11

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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- ◇ *Try to finish whole questions even if you cannot do many.*
- ◇ *You will have done well if you hand in full solutions to two or more questions.*
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1. A regular polygon P has four more sides than another regular polygon Q, and their interior angles differ by 1°. How many sides does P have?

2. Hudson labels each of the four vertices of a triangular pyramid with a different integer chosen from 1 to 15. For each of the four triangular faces, he then calculates the mean of the three numbers at the vertices of the face.

Given that the means calculated by Hudson are all integers, how many different sets of four numbers could he have chosen to label the vertices of the triangular pyramid?

3. It is possible to write $15129 = 123^2$ as the sum of three distinct squares: $15129 = 27^2 + 72^2 + 96^2$.

(i) By using the identity $(a+b)^2 \equiv a^2+2ab+b^2$, or otherwise, find another way to write 15129 as the sum of three distinct squares.

(ii) Hence, or otherwise, show that $378225 = 615^2$ can be written as the sum of six distinct squares.

4. Mr Evans has a class containing an even number of students. He calculated that in the end-of-term examination the mean mark of the students was 58, the median mark was 80 and the difference between the lowest mark and the highest mark was 40. Show that Mr Evans made a mistake in his calculations.

5. A square $ABCD$ has side-length 2, and M is the midpoint of BC. The circle S inside the quadrilateral $AMCD$ touches the three sides AM, CD and DA. What is its radius?

6. A robot sits at zero on a number line. Each second the robot chooses a direction, left or right, and at the sth second the robot moves 2^{s-1} units in that direction on the number line.

For which integers n are there infinitely many routes the robot can take to reach n?

(You may use the fact that every positive integer can be written as a sum of different powers of 2. For example, $19 = 2^0 + 2^1 + 2^4$*)*