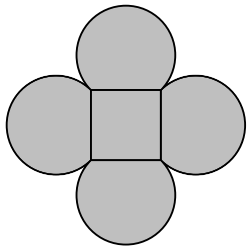
Solutions to the 2017 Olympiad Hamilton Paper

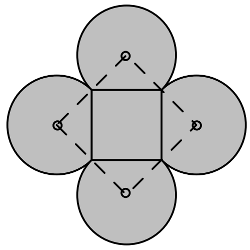
H1. The diagram shows four equal arcs placed on the sides of a square. Each arc is a major arc of a circle with radius 1 cm, and each side of the square has length $\sqrt{2}$ cm.



What is the area of the shaded region?

Solution

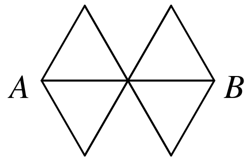
Join the centres of the circles, as shown in the diagram alongside, to form a quadrilateral whose sides have length 1 cm + 1 cm. Each angle of this quadrilateral is equal to 90° from the converse of Pythagoras' Theorem, since we are given the 'inner' square has sides of length $\sqrt{2}$ cm. Because it has equal sides. it follows that the quadrilateral is a square.



The shaded region comprises this square and four sectors of circles, each of radius 1 cm and angle 270°, thus its area is equal to $2 \times 2 + 4 \times \frac{3}{4} \times \pi \times 1^2$, in cm^2.

Therefore, in cm^2, the shaded area is equal to $4 + 3\pi$.

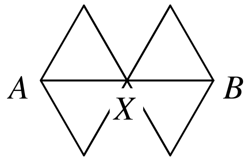
H2. A ladybird walks from A to B along the edges of the network shown. She never walks along the same edge twice. However, she may pass through the same point more than once, though she stops the first time she reaches B.



How many different routes can she take?

Solution

Label the centre point X, as shown in the diagram alongside. Clearly any route that the ladybird takes from A to B passes through X, and she stops the first time she reaches B.



Therefore the number of different routes that the ladybird can take is equal to

(the number of routes from A to X) × (the number from X to B).

The number of routes from A to X is equal to

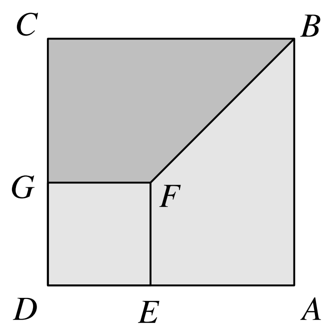
(the number of 'direct' routes from A to X) + (the number from A to X that visit X twice),

which is $3 + 3 \times 2$.

However, the number of routes from X to B is just 3, since the ladybird stops the first time she reaches B, so that it is not possible for her to visit X again. Thus the total number of different routes that the ladybird can take is 9×3, which equals 27.

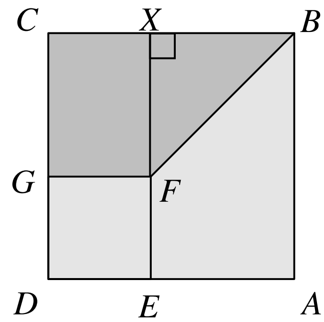
H3. The diagram shows squares *ABCD* and *EFGD*. The length of *BF* is 10 cm. The area of trapezium *BCGF* is 35 cm^2.

What is the length of *AB*?



Solution

The point F lies on the diagonal BD of the square $ABCD$, so that $\angle FBC$ is equal to 45°. Let point X lie on BC so that $\angle FXB = 90°$, as shown in the diagram alongside. Then $\angle XFB = 45°$ from the angle sum of triangle BXF; it follows from 'sides opposite equal angles are equal' that $BX = XF$.



Now, using Pythagoras' Theorem in triangle BXF, we obtain $XF = 5\sqrt{2}$ cm. But $CGFX$ is a rectangle, so that $CG = XF$.

Let the length of AB be a cm. Then

$\frac{1}{2} \times 5\sqrt{2} \times (2a - 5\sqrt{2}) = 35$ and so $2a - 5\sqrt{2} = \dfrac{14}{\sqrt{2}}$.

Hence $2a = 5\sqrt{2} + 14\dfrac{\sqrt{2}}{2}$, so that $a = 6\sqrt{2}$.

Therefore the length of AB is $6\sqrt{2}$ cm.

H4. The largest of four different real numbers is d. When the numbers are summed in pairs, the four largest sums are 9, 10, 12 and 13.

What are the possible values of d?

Solution

Let the other three different numbers be a, b and c, in increasing order. Then each of them is less than d, so that $c + d$ is the largest sum of a pair. The next largest is $b + d$, because it is larger than any other sum of a pair. But we do not know whether $b + c$ or $a + d$ is next (though each of these is larger than $a + c$, which in turn is larger than $a + b$). There are thus two cases to deal with, depending on whether $b + c \leqslant a + d$ or $a + d < b + c$.

$\boldsymbol{b + c \leqslant a + d}$

We have

$$b + c = 9, \tag{1}$$

$$a + d = 10, \tag{2}$$

$$b + d = 12 \tag{3}$$

$$\text{and} \quad c + d = 13. \tag{4}$$

From equations (1), (3) and (4), we find that $2d = 16$, so that $d = 8$.

$\boldsymbol{a + d < b + c}$

We have

$$a + d = 9, \tag{5}$$

$$b + c = 10, \tag{6}$$

$$b + d = 12 \tag{7}$$

$$\text{and} \quad c + d = 13. \tag{8}$$

From equations (6) to (8), we find that $2d = 15$, so that $d = 7.5$.

In each case, it is possible to find the values of a, b and c from the equations, and to check that these fit the conditions in the question.

Therefore the possible values of d are 7.5 and 8.

H5. In the trapezium $ABCD$, the lines AB and DC are parallel, $BC = AD$, $DC = 2AD$ and $AB = 3AD$.

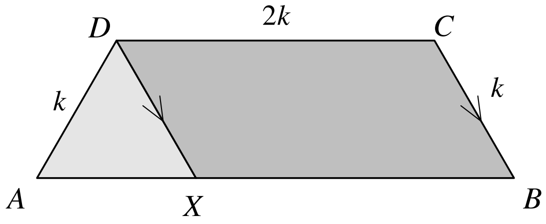
The angle bisectors of $\angle DAB$ and $\angle CBA$ intersect at the point E.

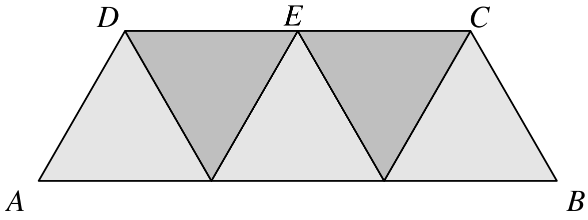
What fraction of the area of the trapezium $ABCD$ is the area of the triangle ABE?

Solution

Let $BC = AD = k$, so that $DC = 2k$ and $AB = 3k$, and let the point X lie on AB so that $XBCD$ is a parallelogram, as shown in the diagram on the left below. It follows that $DX = k$ and $XB = 2k$ (opposite sides of a parallelogram), so that $AX = k$.

Hence triangle AXD has three equal sides—it is therefore an equilateral triangle. In particular, this means that angle DAX is equal to 60°.





As a consequence, the trapezium $ABCD$ is actually made up from five equilateral triangles, as shown in the diagram on the right above.

Now the triangle ABE comprises one equilateral triangle and two half-rhombuses. The area of the two half-rhombuses is equal to the area of two equilateral triangles.

Therefore the area of the triangle ABE is $\frac{3}{5}$ of the area of the trapezium $ABCD$.

H6. Solve the pair of simultaneous equations

$$x^2 + 3y = 10 \qquad \text{and}$$

$$3 + y = \frac{10}{x}.$$

Solution

First, let us number the two given equations, so that it is easy to refer to them.

$$x^2 + 3y = 10 \tag{1}$$

$$3 + y = \frac{10}{x} \tag{2}$$

It is possible to eliminate one of the two unknowns by substituting from equation (2) into equation (1), but this leads to a cubic equation. We present another method that avoids this.

By subtracting $x \times$ equation (2) from equation (1), we get

$$x^2 + 3y - 3x - xy = 0$$

so that

$$(x - 3)(x - y) = 0.$$

Hence either $x = 3$ or $x = y$. We deal with each of these two cases separately.

$\boldsymbol{x = 3}$

Using equation (1), say, we obtain $y = \frac{1}{3}$.

$\boldsymbol{x = y}$

Using equation (1) we obtain

$$x^2 + 3x = 10$$

so that

$$x^2 + 3x - 10 = 0.$$

Hence

$$(x - 2)(x + 5) = 0,$$

and therefore either $x = 2$ or $x = -5$. When $x = 2$ then $y = 2$; when $x = -5$ then $y = -5$.

By checking in the two given equations, we find that all three solutions are valid. Thus there are three solutions of the simultaneous equations, namely

$$\text{either } x = -5 \text{ and } y = -5,$$
$$\text{or } x = 2 \text{ and } y = 2,$$
$$\text{or } x = 3 \text{ and } y = \tfrac{1}{3}.$$