United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Hamilton paper

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. Susie thinks of a positive integer n. She notices that, when she divides 2023 by n, she is left with a remainder of 43. Find how many possible values of n there are.

Solution

Since Susie's number leaves a remainder of 43 when dividing it into 2023, we know that

$$2023 = kn + 43$$

where k is an integer. Also, note that $n > 43$, because the remainder needs to be smaller than the number Susie divides by.

Hence $kn = 1980$, so Susie's number could be any factor of 1980 which is larger than 43.

$1980 = 2^2 \times 3^2 \times 5 \times 11$.

Now we could just write a list of all the factors of 1980, but it's a little quicker to recognise that, aside from 44×45, each pair of factors of 1980 will contribute exactly one possible value of n.

We can count the total number of factors by product of the numbers one higher than each of the powers of the prime factorisation. The total number of factors of 1980 is $(2+1)(2+1)(1+1)(1+1) = 36$. In each pair apart from 44 and 45, one is above 43. This leads to the total number of possible values of n being 19.

For reference although not required by the question, those numbers are 1980, 990, 660, 495, 396, 330, 220, 198, 180, 165, 132, 110, 99, 90, 66, 60, 55, 45 and 44.

2. The two positive integers a, b with $a > b$ are such that a% of b% of a and b% of a% of b differ by 0.003. Find all possible pairs (a, b).

Solution

a% of b% of a is $\frac{a}{100} \times \frac{b}{100} \times a = \frac{a^2b}{10000}$. Similarly, b% of a% of b is $\frac{ab^2}{10000}$.

This leads to

$$\frac{a^2b - ab^2}{10000} = 0.003$$
$$ab(a-b) = 30$$

Since a and b are integers, we know a, b and $a-b$ are all factors of 30.

Further, we know that $a < 10$, since if $a \geq 10$ then one of b and $a - b \geq 5$, which would make the product too big.

We also know that a (being the largest of the three numbers a, b and $a-b$) must be greater than $\sqrt[3]{30} > \sqrt[3]{27} = 3$.
Hence a, being a factor of 30 between 3 and 10, can only be 5 or 6.

If $a = 5$, then

$$b(5-b) = 6$$
$$b^2 - 5b + 6 = 0$$
$$(b-2)(b-3) = 0$$
$$b = 2 \text{ or } b = 3$$

If instead $a = 6$, then

$$b(6-b) = 5$$
$$b^2 - 6b + 5 = 0$$
$$(b-1)(b-5) = 0$$
$$b = 1 \text{ or } b = 5$$

Hence, in total, there are four possible pairs of (a, b): $(5, 2)$, $(5, 3)$, $(6, 1)$ and $(6, 5)$.

3. The nth term of a sequence is the first non-zero digit of the decimal expansion of $\frac{1}{\sqrt{n}}$. How many of the first one million terms of the sequence are equal to 1?

SOLUTION

In the first one million terms, $0.001 \leq \dfrac{1}{\sqrt{n}} \leq 1$.

If the first non-zero digit is 1, we have four inequalities which satisfy the question.

$$1 \leq \frac{1}{\sqrt{n}} < 2 \tag{1}$$
$$0.1 \leq \frac{1}{\sqrt{n}} < 0.2 \tag{2}$$
$$0.01 \leq \frac{1}{\sqrt{n}} < 0.02 \tag{3}$$
$$0.001 \leq \frac{1}{\sqrt{n}} < 0.002 \tag{4}$$

Inequality (1) gives $\dfrac{1}{4} < n \leq 1$.

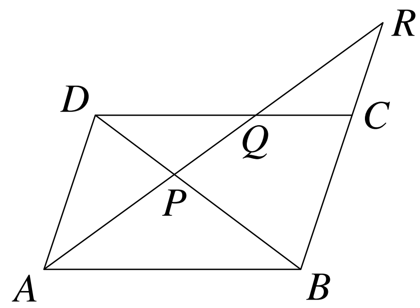
Inequality (2) gives $25 < n \leq 100$.

Inequality (3) gives $2500 < n \leq 10000$.

Inequality (4) gives $250000 < n \leq 1000000$.

Adding up the number of integers in each solution set gives the total number of solutions as $1 + 75 + 7500 + 750000 = 757576$.

4. In the parallelogram $ABCD$, a line through A meets BD at P, CD at Q and BC extended at R. Prove that $\dfrac{PQ}{PR} = \left(\dfrac{PD}{PB}\right)^2$.



Solution

$\angle PAB = \angle PQD$ by alternate angles.

$\angle PBA = \angle PDQ$ by alternate angles.

Therefore, triangles PAB and PQD are similar because they have the same angles.

So $\dfrac{PA}{PQ} = \dfrac{PB}{PD}$.

$\angle PAD = \angle PRB$ by alternate angles.

$\angle PDA = \angle PBR$ by alternate angles.

Therefore, triangles PAD and PRB are similar because they have the same angles.

So $\dfrac{PA}{PD} = \dfrac{PR}{PB}$.

These equations give us $PA = \dfrac{PR \times PD}{PB} = \dfrac{PB \times PQ}{PD}$.

This rearranges to give $\dfrac{PQ}{PR} = \left(\dfrac{PD}{PB}\right)^2$.

5. Mickey writes down on a board n consecutive whole numbers the smallest of which is 2023. He then replaces the largest two numbers on the board with their difference, reducing the number of numbers on the board by one. He does this repeatedly until there is only a single number on the board. For which values of n is this last remaining number 0?

Solution

Note that subtracting two numbers and replacing by the difference does not change the parity of the sum of the numbers: If both numbers are odd or both numbers are even then both their sum and their difference are even; if one is odd and one is even then both their sum and their difference are odd.

This means if the total sum originally is odd, the last number must be odd and therefore cannot be 0. This happens whenever n is 1 more than a multiple of 4 or 2 more than a multiple of 4, so these values of n cannot leave 0 as the remaining number.

The largest two numbers on the board are consecutive, so are replaced by 1. Since 1 is never going to be larger than any number already on the board, the remaining largest two numbers will again be consecutive and be replaced by 1 until there are not two of the original list to remove.

If n is a multiple of 4, all the original numbers will have been replaced by an even number of 1's left on the board. These will all be removed in pairs and replaced by 0's. The difference between the 0's will be 0's, so the last number on the board will be 0.

If n is three more than a multiple of 4, all the original numbers except 2023 will have been replaced by an odd number of 1's left on the board. Whenever the board contains a number of 1's and a larger number, k, the larger number, k, and a 1 are replaced by $k - 1$ until there are only 1's left or there is only one number left. If $n < 4047$, there will be fewer than 2023 1's left on the board with 2023, so there will be one number greater than 1 left on the board when all the 1's have been removed. If $n \geq 4047$, there will be at least 2023 1's left with 2023, so the board will get down to only 1's. As explained above, the last number must be even and cannot be greater than 1, so it must end with a 0.

The possible values of n are all multiples of 4 and all numbers at least 4047 which are three more than a multiple of 4.

6. Find all triples (m, n, p) which satisfy the equation

$$p^n + 3600 = m^2$$

where p is prime and m, n are positive integers.

SOLUTION

Rearranging the equation we get $p^n = m^2 - 3600 = (m - 60)(m + 60)$.

Clearly, both $(m - 60)$ and $(m + 60)$ must be factors of p^n. Since p is prime, the only factors of p^n are 1 and integer powers of p, so both brackets can only be 1 or an integer power of p.

If $m - 60 = 1$, then $m + 60 = 121$, giving the solution $p = 11$, $n = 2$ and $m = 61$.

If $m - 60 \neq 1$, then p divides $m - 60$ and $m + 60$, so p divides the difference between them, 120. Therefore p can be only 2, 3 or 5.

If $p = 2$, we are looking for two powers of 2 which differ by 120. Since the larger one must be at least 120 and beyond 128 the difference between powers of 2 is larger than 120, the only solution is $m + 60 = 128$ and $m - 60 = 8$. This gives a solution of $p = 2$, $n = 10$ and $m = 68$.

If $p = 3$, we are looking for two powers of 3 which differ by 120. Since the larger one must be at least 120 and beyond 81 the difference between powers of 3 is larger than 120, there is no solution in this case.

If $p = 5$, we are looking for two powers of 5 which differ by 120. Since the larger one must be at least 120 and beyond 125 the difference between powers of 5 is larger than 120, the only solution is $m + 60 = 125$ and $m - 60 = 5$. This gives a solution of $p = 5$, $n = 4$ and $m = 65$.

The solutions are $(m, n, p) = (61, 2, 11)$, $(65, 4, 5)$ or $(68, 10, 2)$.