Intermediate Mathematical Challenge

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Solutions and investigations

6 February 2020

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	E	B	D	D	C	B	E	E	B	E	B	C	A	A	E	E	C	B	D	B	C	A	A	A

1. What is the value of $2 - (3 - 4) - (5 - 6 - 7)$?

A 11 B 9 C 5 D −5 E −7

SOLUTION **A**

We have

$$\begin{aligned} 2 - (3 - 4) - (5 - 6 - 7) &= 2 - (-1) - (-8) \\ &= 2 + 1 + 8 \\ &= 11. \end{aligned}$$

FOR INVESTIGATION

1.1 What is the value of the following?

(a) $2 - (3 - 4) - (5 - 6 - 7) - (8 - 9 - 10 - 11)$.

(b) $2 - (3 - 4) - (5 - 6 - 7) - (8 - 9 - 10 - 11) - (12 - 13 - 14 - 15 - 16)$.

2. Which one of these is a multiple of 24?

A 200 B 300 C 400 D 500 E 600

SOLUTION **E**

Because $24 = 3 \times 8$, and 3 and 8 have no common factors (other than 1), the integers that are multiples of 24 are the integers that are multiples of both 3 and 8.

It is easy to check that 300 and 600 are multiples of 3, and that 200, 400 and 500 are not multiples of 3.

We also note that $200 = 8 \times 25$ and therefore is multiple of 8. Similarly, it easy to check that 400 and 600 are multiples of 8, but 300 and 500 are not.

We now see that 600 is a multiple of both 3 and 8, and is therefore a multiple of 24, but none of the other options is a multiple of 24.

FOR INVESTIGATION

2.1 Check that 200, 400 and 500 are not multiples of 3.

2.2 Check that 400 and 600 are multiples of 8, but 300 and 500 are not.

2.3 Find the smallest integer greater than 600 that is a multiple of 24.

2.4 Find the smallest integer greater than 1000 that is a multiple of 24.

2.5 Find the smallest integer greater than 1 000 000 that is a multiple of 24.

3. What is the difference between 25% of £37 and 25% of £17?

A £4.25 B £5 C £6 D £7.50 E £9.25

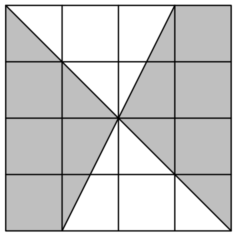
SOLUTION **B**

Because 25% of a number is one-quarter of it, we have

$$\begin{aligned}25\% \text{ of } £37 - 25\% \text{ of } £17 &= £\left(\tfrac{1}{4}(37)\right) - £\left(\tfrac{1}{4}(17)\right)\\ &= £\left(\tfrac{1}{4}(37-17)\right)\\ &= £\left(\tfrac{1}{4}(20)\right)\\ &= £5.\end{aligned}$$

4. What fraction of this diagram is shaded?

A $\frac{13}{32}$ B $\frac{1}{2}$ C $\frac{9}{16}$ D $\frac{5}{8}$ E $\frac{13}{16}$



SOLUTION **D**

We choose units so that each of the 16 small squares in the diagram has sides of length 1 unit. Hence the area of the grid shown in the diagram is 16 square units.

The shaded region is made up of the whole grid with the two unshaded triangles removed.

Each of the unshaded triangles has a base of length 3 and height of length 2. Therefore using the formula

$$\text{area} = \text{base} \times \text{height}$$

for the area of a triangle, we see that the area of each of these triangles is given by

$$\tfrac{1}{2}(3 \times 2) = 3 \text{ square units.}$$

Therefore the total area of the unshaded triangles is 2×3 square units, that is, 6 square units.

Hence the shaded area is 16 square units − 6 square units, that is, 10 square units.

Therefore the fraction of the diagram that is shaded is $\dfrac{10}{16}$.

By cancelling the common factor 2 in the numerator and denominator, we see that this fraction is equal to $\dfrac{5}{8}$.

Note: This is a case where it is easier to calculate the unshaded area, rather than calculate the shaded area directly. This frequently turns out to be a good method. It is used again in the solution to Question 21.

5. Four of the following coordinate pairs are the corners of a square.
Which is the odd one out?

A (4, 1) B (2, 4) C (5, 6) D (3, 5) E (7, 3)

Solution **D**

Commentary

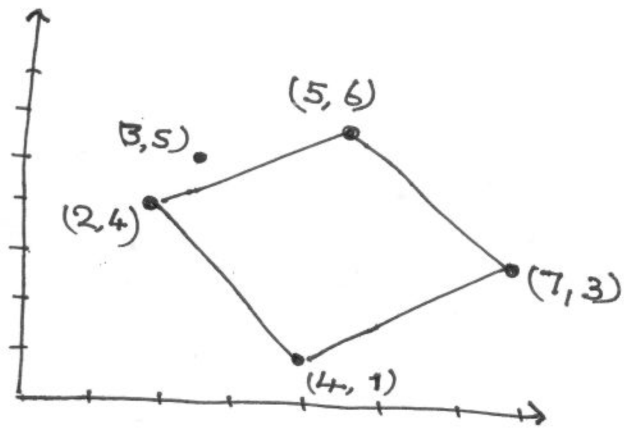
In the context of the IMC the quickest and easiest way to answer this question is to draw a sketch. This is the method used here. Even a rough sketch, drawn freehand, will be good enough to spot the odd point out, as shown below.

However, this method would not be acceptable when a complete mathematical solution is required.

Problems 5.1 and 5.2 cover alternative methods for answering this question using calculations.

From the sketch alongside it is clear that the points with coordinates $(4, 1)$, $(2, 4)$, $(5, 6)$ and $(7, 3)$ are the vertices of the square.

Hence $(3, 5)$ is the odd one out.



For investigation

5.1 (a) Calculate the distance between each pair of the points given in this question.

(b) Deduce that the point with coordinates $(3, 5)$ is the odd point out.

5.2 (a) Calculate the slope of the line joining $(2, 4)$ and $(5, 6)$, and the slope of the line joining $(5, 6)$ and $(7, 3)$.

(b) Deduce that these lines are at right angles to each other.

[Hint: the criterion for lines with slopes m and n to be at right angles is that $m \times n = -1$.]

(c) Repeat the above calculation for the other edges, and check that the other pairs of adjacent edges of the square meet each other at right angles.

(d) Calculate the slopes of the lines joining the point with coordinates $(3, 5)$ to each of the other points. Hence show that no two of these lines meet each other at right angles.

(e) Deduce that the point with coordinates $(3, 5)$ is the odd one out.

6. Which of the following has the largest value?

A 2^6 B 3^5 C 4^4 D 5^3 E 6^2

SOLUTION **C**

COMMENTARY

A natural first thought is that it might be possible to find the expression with the largest value without evaluating each expression. Instead we could use properties of indices.

Using this idea we see that $2^6 = (2^2)^3 = 4^3 < 4^4$, and $6^2 < 8^2 = (2^3)^2 = 2^6$. This shows that neither option A nor option E has the largest value. Unfortunately, however, the values of 3^5 and 4^4 are so close that the only straightforward way to decide which is the larger is to evaluate both. So in the end, it turns out that the best way to answer this question is to evaluate each of the five options.

We have

$$\begin{aligned} 2^6 &= 64, \\ 3^5 &= 243, \\ 4^4 &= 256, \\ 5^3 &= 125, \\ \text{and} \quad 6^2 &= 36. \end{aligned}$$

It follows that the largest of the given options is 4^4.

FOR INVESTIGATION

6.1 Evaluate 2^n for all integer value of n in the range from 1 to 10. [It is useful to have these values of the first ten powers of 2 in your head.]

6.2 Determine for which positive integers n, with $1 \leq n \leq 10$, the integer $2^n - 1$ is prime. What do you notice?

NOTE

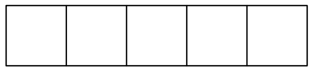
Prime numbers of the form $2^n - 1$ are called *Mersenne primes*. Most computer searches for large prime numbers concentrate on looking for possible Mersenne primes. At the time of writing the largest known prime number, discovered in December 2018, is the Mersenne prime $2^{82\,589\,933} - 1$.

It is not known whether there are infinitely many Mersenne primes. There are 51 known Mersenne primes of which the largest is that given above.

6.3 Mersenne primes are particularly interesting because of their connection with *perfect numbers*. Find out what perfect numbers are and their connection with Mersenne primes.

6.4 From the answer to Problem 6.1 we see that 2^{10} is approximately 1000. Use this approximation to estimate the number of digits in the largest known prime $2^{82\,589\,933} - 1$ when it is written in standard form.

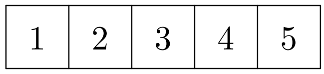
7. Kartik wants to shade three of the squares in this grid blue and Lucy wants to shade the remaining two squares red. There are ten possible finished grids.
In how many of the finished grids are Lucy's red squares next to each other?



A 3 B 4 C 5 D 6 E 8

Solution **B**

In the diagram we have labelled the squares with numbers so that we can refer to them.



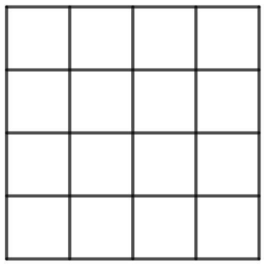
If Lucy colours red two squares that are next to each other, the leftmost of the two squares can be any of the squares 1, 2, 3 and 4, but not 5.

So Lucy's red squares are next to each other in 4 of the finished grids.

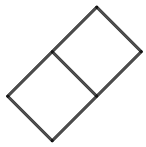
For investigation

7.1 Note that colouring two squares that are next to each other red is the same as placing a 2×1 tile on the grid so that it cover two squares exactly.

How many different positions are there for a 2×1 tile on a 4×4 grid so that it covers two squares exactly?



4×4 grid



2×1 tile

7.2 How many different positions are there for a 2×1 tile on an 8×8 grid so that it cover two squares exactly?

7.3 Suppose that n is an integer with $n > 1$.

Find a formula, in terms of n, for the number of different positions for a 2×1 tile on an $n \times n$ grid so that it covers two squares exactly.

7.4 Suppose that c, d, m and n are positive integers such that $c \leq d \leq m \leq n$.

How many different positions are there for a $c \times d$ tile on an $m \times n$ grid so that it covers cd squares exactly?

8. One of these options gives the value of $17^2 + 19^2 + 23^2 + 29^2$. Which is it?

A 2004 B 2008 C 2012 D 2016 E 2020

Solution **E**

Commentary

The IMC is a *non-calculator* paper. This should be a clue that there is a way to answer this question without working out all the four squares 17^2, 19^2, 23^2 and 29^2 separately and then adding them up.

A better method is to think about the last digit (otherwise known as the *units digit* or *ones digit*) of the numbers involved. It will be seen that this rapidly leads to the correct answer. A more sophisticated way to describe this method is to say that we are using *modular arithmetic* and we are working mod 10.

Because $7^2 = 49$, the last digit of 17^2 is 9. Similarly the last digits of 19^2, 23^2 and 29^2 are 1, 9 and 1, respectively.

It follows that the last digit of $17^2 + 19^2 + 23^2 + 29^2$ is the same as the last digit of $9 + 1 + 9 + 1$. Now $9+1+1+9 = 20$. We can therefore conclude that the last digit of $17^2+19^2+23^2+29^2$ is 0. Hence, given that one of the options is correct, we can deduce that $17^2 + 19^2 + 23^2 + 29^2 = 2020$.

Note: We see from this question that 2020 is the sum of the squares of four consecutive prime numbers. Integers of this form are naturally quite rare. However it was proved in 1770 by Joseph-Louis Lagrange that every positive integer is the sum of at most four squares of positive integers. The proof of this theorem, whose 250th anniversary occurs this year, is not easy.

For investigation

8.1 Investigate the biography of Lagrange.

8.2 Calculate the values of 17^2, 19^2, 23^2 and 29^2, and check that their sum is 2020.

8.3 Which is the smallest positive integer which *cannot* be expressed as the sum of fewer than four squares of positive integers.

8.4 Which is the smallest positive integer, greater than 2020, that is the sum of the squares of four consecutive primes?

8.5 The number 2020 may be written as the sum of the squares of two positive integers in two different ways. Find them. (*Note*: We are not concerned with the order of the squares. For example $2^2 + 3^2$ and $3^2 + 2^2$ count as the same way of writing 13 as the sum of two squares.)

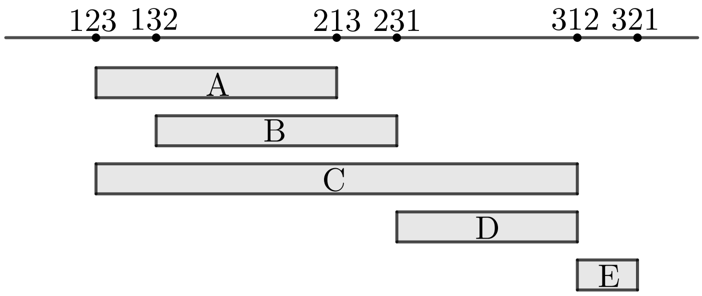
8.6 The number 2020 may be written as the sum of the squares of three positive integers in just one way. Find it.

8.7 It turns out 2020 may be written as the sum of the squares of four positive integers in no fewer than 48 different ways. It is *not* recommended that you try to find these by hand. However, if you are familiar with computer programming, write a program to list these 48 different expressions of 2020 as the sum of four squares.

9. Adam's house number is in exactly one of the following ranges. Which one?

A 123 to 213 B 132 to 231 C 123 to 312 D 231 to 312 E 312 to 321

Solution **E**



The diagram shows the different ranges on the number line. For clarity, the diagram is not to scale.

We see that the only range that contains numbers not in any other range is the range corresponding to option E. This is the range from 312 to 321.

10. What is the value of $\dfrac{2468 \times 2468}{2468 + 2468}$?

A 2 B 1234 C 2468 D 4936 E 6 091 024

Solution **B**

> Commentary
>
> As with Question 8, you should look for a method which avoids too much arithmetic. Without the use of the calculator, you should avoid working out 2468×2468 and $2468 + 2468$ separately, and then doing a long division. Instead, the best method is to cancel the common factor 2468 in the numerator and the denominator.

We have

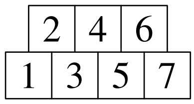
$$\frac{2468 \times 2468}{2468 + 2468} = \frac{2468 \times 2468}{2 \times 2468} = \frac{2468}{2} = 1234.$$

For investigation

10.1 What is the value of

$$\frac{2468 \times 2468 \times 2468}{2468 \times 2468 + 2468 \times 2468}\ ?$$

11. I start at square "1", and have to finish at square "7", moving at each step to a higher numbered adjacent square.
How many possible routes are there?



A 7 B 9 C 10 D 11 E 13

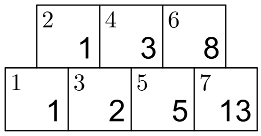
SOLUTION **E**

COMMENTARY

We adopt a method which can be used in many similar problems which ask you to count the number of different routes from a square S to a square T. It works whenever the rules mean that you cannot visit a square more than once.

There is just one way to reach the initial square S, namely by starting there. The idea is to count the number of routes ending at each other square, according to the following principle. The number of ways to reach another square X is the sum of the number of ways of reaching all the squares from where you can get to X in just one move.

We have used the method explained in the above commentary to produce the diagram shown alongside.



In the squares in this diagram the smaller numbers in the top left-hand corners are the numbers that label the squares in the diagram in the question.

The larger numbers in the bottom right-hand corners are the number of ways of reaching the given square along routes allowed by the question. These numbers are calculated according to the method described in the above commentary.

For example square the square "6" can be reached in one move from both "4" and "5". The number of ways of reaching "4" is 3, and the number of ways of reaching "5" is 5. So the number of ways of reaching square "6" is 3 + 5 = 8.

We therefore conclude that the number of possible routes to square "7" is 13.

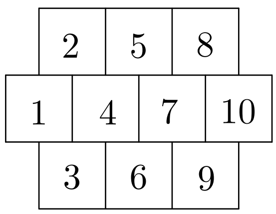
FOR INVESTIGATION

11.1 Check that the numbers in the diagram in the above solution are correct.

11.2 What do you notice about the numbers that count the number of ways of reaching the different squares? Explain why they have the pattern that they do have.

11.3 I start at square "1", in the diagram alongside, and have to finish at square "10", moving at each step to a higher numbered adjacent square.

How many possible routes are there?



12. Farmer Fatima rears chickens and goats. Today she returned from market and said, "I sold 80 animals, and now there are 200 fewer legs on my farm than before!"
How many goats did she sell?

A 15 B 20 C 25 D 30 E 35

Solution **B**

Let g be the number of goats that Farmer Fatima sold, and let c be the number of chickens that she sold.

Because Farmer Fatima sold 80 animals,

$$g + c = 80. \qquad (1)$$

Goats have four legs and chickens two legs. Therefore, because after the sale there were 200 fewer legs on the farm,

$$4g + 2c = 200. \qquad (2)$$

If we multiply equation (1) by 2, we obtain

$$2g + 2c = 160. \qquad (3)$$

Subtracting equation (3) from equation (2) we obtain

$$2g = 40,$$

and therefore

$$g = 20.$$

Therefore Farmer Fatima sold 20 goats.

For investigation

12.1 How many chickens did Farmer Fatima sell?

12.2 After saying that there were 200 fewer legs on the farm, Farmer Fatima added, "Now I have 100 animals on the farm, and between them they have 266 legs."

How many chickens and how many goats does Farmer Fatima now have?

12.3 An alternative way to solve equations (1) and (2), is to rewrite equation (1) as $c = 80 - g$ and then use this to substitute for c in equation (2). Use this method to solve the original equations, that is, equations (1) and (2) above.

13. What is half of 1.6×10^6?

A 8×5^6 B 4×10^6 C 8×10^5 D 8×10^2 E 1.6×10^3

Solution **C**

Half of 1.6 is 0.8, which can be written as $\frac{8}{10}$. Therefore half of 1.6×10^6 is $\frac{8}{10} \times 10^6$ which is equal to 8×10^5.

14. The result of the calculation $9 \times 11 \times 13 \times 15 \times 17$ is the six-digit number '$3n8185$'. What is the value of n?

A 2 B 4 C 6 D 8 E 0

SOLUTION **A**

COMMENTARY

We are not asked to find the value of the product, but only the value of the digit n. Therefore we look for a method which avoids the need to do four multiplications to evaluate $9 \times 11 \times 13 \times 15 \times 17$.

METHOD 1

The number $9 \times 11 \times 13 \times 15 \times 17$ is a multiple of 9.

The criterion for an integer to be a multiple of 9 is that the sum of its digits is a multiple of 9.

The sum of the digits of '$3n8185$' is $3 + n + 8 + 1 + 8 + 5$ which is equal to $25 + n$. Now n is a digit in the range from 0 to 9, and so $25 + n$ is in the range 25 to 34. The only multiple of 9 in this range is 27, which occurs when n is 2.

METHOD 2

The number $9 \times 11 \times 13 \times 15 \times 17$ is a multiple of 11. The criterion for an integer to be a multiple of 11 is that the difference between the sum of its digits in odd places, counting from the left, and the sum of its digits in even places is a multiple of 11.

The sum of the digits of '$3n8185$' in odd places is $3 + 8 + 8$ which is equal to 19. The sum of its digits in even places is $n + 1 + 5$, which is equal to $6 + n$.

The difference between these sums is

$$19 - (6 + n) = 13 - n.$$

Since n is in the range from 0 to 9, $13 - n$ is in the range from 4 to 13. The only multiple of 11 in this range is 11, which occurs when n is 2.

FOR INVESTIGATION

14.1 Prove that the criterion for an integer to be a multiple of 9 is that the sum of its digits is a multiple of 9.

14.2 Prove that the criterion for an integer to be divisible by 11 is that the difference between the sum of its digits in odd places and the sum of its digits in even places, is a multiple of 11.

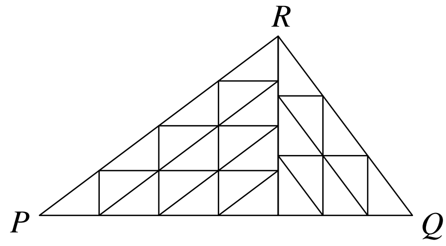
14.3 Is the integer 123 456 789 a multiple of 9?

14.4 Find the smallest integer which is greater than 123 456 789 and which is a multiple of 11.

15. Triangle PQR has been divided into twenty-five congruent right-angled triangles, as shown. The length of RP is 2.4 cm.

What is the length of PQ?

A 3 cm B 3.2 cm C 3.6 cm
D 4 cm E 4.8 cm



Solution **A**

We note first that $\angle QPR$ and $\angle RQP$ are equal to the two non-right-angles of each of the twenty-five congruent smaller triangles into which the triangle PQR is divided.

Since the triangle PQR shares two of the angles of the smaller triangles, it follows that the triangle PQR is similar to each of these smaller triangles. In particular, $\angle QRP$ is a right angle.

From the diagram in the question we see that the length of RP is four times the length of the hypotenuse of the smaller triangles, and the length of RQ is three times the length of the hypotenuse of the smaller triangles.

It follows that $RP : RQ = 4 : 3$. Therefore, because $\angle QRP = 90°$, we may deduce that $PQ : RP : RQ = 5 : 4 : 3$.

Now RP has length 2.4 cm. It follows that the length of PQ is given by

$$PQ = \frac{5}{4} \times RP = \frac{5}{4} \times 2.4\,\text{cm} = \frac{5 \times 2.4}{4}\,\text{cm} = \frac{12}{4}\,\text{cm} = 3\,\text{cm}.$$

For investigation

15.1 Explain why it follows from the fact that two of the angles of the triangle PQR are equal to two of the angles of the smaller triangles that PQR is similar to the smaller triangles.

15.2 The solution above assumes that if the two shorter sides of a right-angled triangle are in the ratio 4 : 3, then the three sides of the triangle are in the ratio 5 : 4 : 3. From which theorem does this follow?

15.3 The numbers 5, 4, 3 form what is called a *primitive Pythagorean triple*, because they are integers with no common factor which are the side lengths of a right-angled triangle. Find some more examples of primitive Pythagorean triples.

16. As a decimal, what is the value of $\frac{1}{9} + \frac{1}{11}$?

A 0.10 B 0.20 C 0.2020 D 0.202 020 E $0.\dot{2}\dot{0}$

Solution **E**

Commentary

We give two methods for tackling this problem.

In the first method we begin by working out the sum as a rational number. Then we convert the answer to a decimal.

In the second method we convert $\frac{1}{9}$ and $\frac{1}{11}$ to decimals and then add them.

Method 1

We have

$$\frac{1}{9} + \frac{1}{11} = \frac{11 + 9}{99} = \frac{20}{99}.$$

$$\begin{array}{r l} & 0.2020\ldots \\ 99 & \overline{\big)20.0000\ldots} \\ & \underline{0} \\ & 20\,0 \\ & \underline{19\,8} \\ & 20 \\ & \underline{0} \\ & 200 \\ & \underline{198} \\ & 20 \end{array}$$

From the division sum alongside we see that $\frac{20}{99}$ as a decimal is the recurring decimal 0.2020....

This is the recurring decimal which is written as,

$$0.\dot{2}\dot{0}.$$

Method 2

We have

$$\frac{1}{9} + \frac{1}{11} = 0.111\,111\cdots + 0.090\,090\ \cdots = 0.202\,020\cdots = 0.\dot{2}\dot{0}.$$

For investigation

16.1 Write the following decimals as fractions in their simplest form, that is, in the form $\frac{p}{q}$, where p and q are positive integers with no common factor other than 1.

(a) 0.10, (b) 0.20, (c) 0.2020 (d) 0.202 202.

16.2 Write the following fractions as decimals.

(a) $\frac{1}{8}+\frac{1}{9}$, (b) $\frac{1}{7}+\frac{1}{8}$, (c) $\frac{1}{7}+\frac{1}{9}$, (d) $\frac{1}{7}+\frac{1}{13}$.

16.3 Write the recurring decimal $0.\dot{1}2\dot{3}$ as a fraction in its simplest terms.

[*Hint*: In case you have not seen the method for converting a recurring decimal into a fraction, we give an example here.

We consider the recurring decimal $0.\dot{7}\dot{2}$. We let

$$x = 0.\dot{7}\dot{2}.$$

We first note that we could write x as

$$x = 0.727272\ldots.$$

We multiply both sides of the last equation by 100 to obtain

$$100x = 72.727272\ldots.$$

That is

$$100x = 72 + x.$$

From the last equation it follows that

$$99x = 72,$$

and hence

$$x = \frac{72}{99} = \frac{8}{11}.]$$

17. The Knave of Hearts stole some tarts. He ate half of them, and half a tart more. The Knave of Diamonds ate half of what was left, and half a tart more. Then the Knave of Clubs ate half of what remained, and half a tart more. This left just one tart for the Knave of Spades.
How many tarts did the Knave of Hearts steal?

A 63 B 31 C 19 D 17 E 15

SOLUTION **E**

COMMENTARY

In a problem of this type, a feasible method is to work backwards from the given options. The disadvantage of this method is that you cannot be sure how much work is involved. If the correct option is E, you may need to do five calculations before finding the right answer. However, in this case, although the correct option is E, if you begin with option A and notice the pattern, you can save the need to do all five calculations.

An alternative method is to think about the effect of eating half of the tarts, and then half a tart more.

Method 1

If there were 63 tarts to start with, the Knave of Hearts would have eaten $\frac{1}{2}(63) + \frac{1}{2}$ tarts, that is, 32 tarts leaving 31. Then the Knave of Diamonds would have eaten $\frac{1}{2}(31) + \frac{1}{2} = 16$ tarts leaving 15. Then the Knave of Clubs would have eaten $\frac{1}{2}(15) + \frac{1}{2} = 8$ tarts, leaving 7 tarts for the Knave of Spades.

Although this means that option A is not correct, we can see the pattern by looking at the numbers involved. We will write "$a \longrightarrow b$" to mean that if, starting with a tarts, you eat half of them and half a tart more, you are left with b tarts. Using this notation, what we have discovered is that

$$63 \longrightarrow 31 \longrightarrow 15 \longrightarrow 7.$$

Is it straightforward to check that this sequence continues as

$$63 \longrightarrow 31 \longrightarrow 15 \longrightarrow 7 \longrightarrow 3 \longrightarrow 1.$$

We now see that if Knave of Hearts begins with 15 tarts, then after the three knaves have eaten their tarts, the Knave of Spades is left with just one tart.

Method 2

We use the notation "$a \longrightarrow b$" as in Method 1.

Now, suppose that $a \longrightarrow b$.

A knave who starts with a tarts and eats half of them and half a tart more is left with

$$a - \tfrac{1}{2}a - \tfrac{1}{2} = \tfrac{1}{2}(a - 1) \text{ tarts.}$$

Therefore, $b = \frac{1}{2}(a - 1)$. Hence $a - 1 = 2b$ and so $a = 2b + 1$. Thus $2b + 1 \longrightarrow b$.

Hence, putting $b = 1, 3$ and 7, we have $3 \longrightarrow 1$, $7 \longrightarrow 3$ and $15 \longrightarrow 7$, respectively. That is

$$15 \longrightarrow 7 \longrightarrow 3 \longrightarrow 1.$$

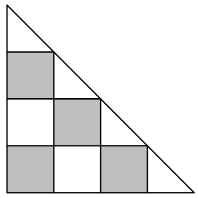
Therefore, if the Knave of Spades is to be left with just one tart, the Knave of Clubs must start with 15 tarts.

For investigation

17.1 Suppose that the Knave of Spades is left with 10 tarts. How many tarts did the Knave of Hearts steal in this case?

17.2 Find a formula, in terms of n, for the number of tarts that the Knave of Hearts needs to steal, if the Knave of Spades is to be left with n tarts.

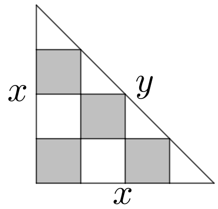
18. The diagram shows an isosceles right-angled triangle which has a hypotenuse of length y. The interior of the triangle is split up into identical squares and congruent isosceles right-angled triangles. What is the total shaded area inside the triangle?



A $\frac{y^2}{2}$ B $\frac{y^2}{4}$ C $\frac{y^2}{8}$ D $\frac{y^2}{16}$ E $\frac{y^2}{32}$

Solution **C**

Let the two equal sides of the large right-angled triangle have length x.



By Pythagoras' Theorem, applied to this triangle, we have $x^2 + x^2 = y^2$.

Therefore

$$x^2 = \frac{y^2}{2}.$$

It follows that the area of the large right-angled triangle is given by

$$\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(x \times x) = \frac{x^2}{2} = \frac{y^2}{4}.$$

The shaded area is made up of four of the small squares.

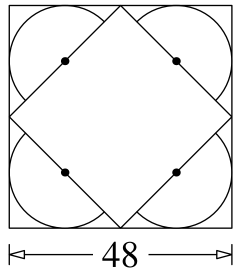
The unshaded area is made up of two of the small squares and four half squares, which is the same as the area of four of the small squares. Hence the shaded area is equal to the unshaded area.

Therefore the shaded area is half the area of the large right-angled triangle.

It follows that the shaded area is given by

$$\frac{1}{2} \times \frac{y^2}{4} = \frac{y^2}{8}.$$

19. The diagram shows two squares and four equal semicircles. The edges of the outer square have length 48 and the inner square joins the midpoints of the edges of the outer square. Each semicircle touches two edges of the outer square, and the diameter of each semicircle lies along an edge of the inner square.

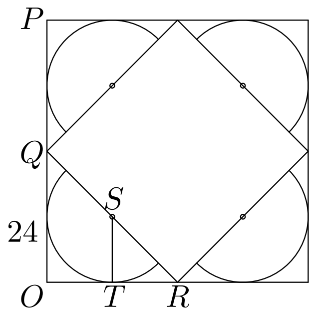


What is the radius of each semicircle?

A 10 B 12 C 14 D 16 E 18

SOLUTION **B**

Let O and P be two vertices of the larger square, let Q and R be the midpoints of two sides of this square, let S be the centre of one of the semicircles and let T be the point where this semicircle touches OR, all as shown in the diagram.



Because ST is a radius, and OR is a tangent to the semicircle at the point T, ST is perpendicular to OR. It follows that the triangles RQO and RST are similar.

Because Q is the midpoint of OP, the length of QO is 24.

We leave it as an exercise to show that S is the midpoint of RQ.

Since the triangles RQO and RST are similar, it follows that

$$\frac{ST}{QO} = \frac{RS}{RQ} = \frac{1}{2}\,.$$

Hence $ST = \frac{1}{2}QO = \frac{1}{2}(24) = 12$.

Therefore the radius of each semicircle is 12.

FOR INVESTIGATION

19.1 Explain why it follows from the fact that ST is perpendicular to OR that the triangles RQO and RST are similar.

19.2 Prove that S is the midpoint of RQ.

20. For any fixed value of x, which of the following four expressions has the largest value?

$(x+1)(x-1)$ $\quad (x+\frac{1}{2})(x-\frac{1}{2})$ $\quad (x+\frac{1}{3})(x-\frac{1}{3})$ $\quad (x+\frac{1}{4})(x-\frac{1}{4})$

A $(x+1)(x-1)$ B $(x+\frac{1}{2})(x-\frac{1}{2})$ C $(x+\frac{1}{3})(x-\frac{1}{3})$
D $(x+\frac{1}{4})(x-\frac{1}{4})$ E it depends on the value of x

Solution **D**

We use the factorization $x^2 - a^2 = (x+a)(x-a)$ of the difference of two squares. This gives

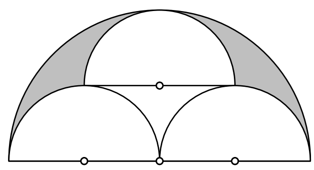
$$(x+1)(x-1) = x^2 - 1^2 = x^2 - 1,$$
$$(x+\tfrac{1}{2})(x-\tfrac{1}{2}) = x^2 - \tfrac{1}{2}^2 = x^2 - \tfrac{1}{4},$$
$$(x+\tfrac{1}{3})(x-\tfrac{1}{3}) = x^2 - \tfrac{1}{3}^2 = x^2 - \tfrac{1}{9},$$
and $$(x+\tfrac{1}{4})(x-\tfrac{1}{4}) = x^2 - \tfrac{1}{4}^2 = x^2 - \tfrac{1}{16}.$$

Now, since $\frac{1}{16} < \frac{1}{9} < \frac{1}{4} < 1$, for each value of x, we have $x^2 - \frac{1}{16} > x^2 - \frac{1}{9} > x^2 - \frac{1}{4} > x^2 - 1$.

That is, $(x+\frac{1}{4})(x-\frac{1}{4}) > (x+\frac{1}{3})(x-\frac{1}{3}) > (x+\frac{1}{2})(x-\frac{1}{2}) > (x+1)(x-1)$.

Therefore, for each value of x, the expression $(x+\frac{1}{4})(x-\frac{1}{4})$ has the largest value.

21. The diagram shows four semicircles, one with radius 2 cm, touching the other three, which have radius 1 cm.

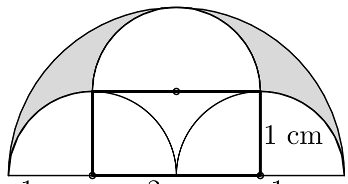


What is the total area, in cm^2, of the shaded regions?

A 1 B $\pi - 2$ C $2\pi - 5$ D $\frac{3}{2}$ E $\frac{1}{2}\pi$

Solution **B**

We see from the diagram alongside that the unshaded region is made up of a 2 cm × 1 cm rectangle, two quarter circles of radius 1 cm, and a semicircle of radius 1 cm. Therefore the area of the unshaded region is $2\,\text{cm}^2$ + the area of a circle of radius 1 cm, that is, $(2 + \pi \times 1^2)\,\text{cm}^2 = (2+\pi)\,\text{cm}^2$.



The total area of the semicircle with radius 2 cm is $\frac{1}{2} \times (\pi \times 2^2)\,\text{cm}^2 = 2\pi\,\text{cm}^2$.

Therefore the total area, in cm^2, of the shaded regions is $2\pi - (2+\pi) = \pi - 2$.

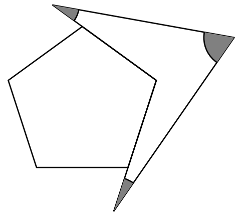
For investigation

21.1 Prove that the rectangle mentioned in the solution really is a rectangle.

22. The diagram shows a regular pentagon and an irregular quadrilateral.

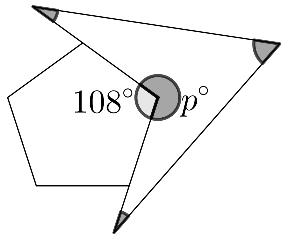
What is the sum of the three marked angles?

A $72°$ B $90°$ C $108°$ D $126°$ E $144°$



SOLUTION **C**

Let the unmarked interior angle of the quadrilateral in the diagram of the question be be $p°$.



Also, let the sum of the three marked angles in the quadrilateral be $x°$.

Each interior angle of a regular pentagon is $108°$. [You are asked to prove this in Problem 22.1.]

Because the sum of the angles at a point is $360°$,

$$p + 108 = 360.$$

Because the sum of the interior angles of a quadrilateral is $360°$,

$$p + x = 360.$$

Therefore

$$p + 108 = 360 = p + x.$$

It follows that $x = 108$.

Therefore the sum of the three marked angles is $108°$.

FOR INVESTIGATION

22.1 Prove that each interior angle of a regular pentagon is $108°$.

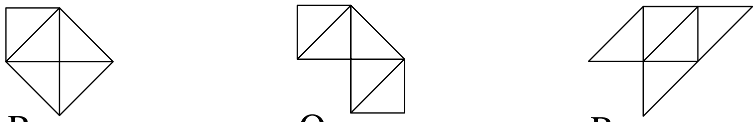
22.2 Prove that the sum of the angles of a quadrilateral is $360°$.

22.3 (a) Find a formula for the sum of the interior angles of a polygon with n vertices, where n is a positive integer with $n > 2$.

(b) Deduce a formula for the interior angle of a regular polygon with n vertices.

(c) Check that your formulas are compatible with your answers to Problems 22.1 and 22.2.

23. Five congruent triangles, each of which is half a square, are placed together edge to edge in three different ways as shown to form shapes P, Q and R.



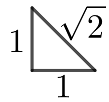
P Q R

Which of the following lists gives the shapes in ascending order of the lengths of their perimeters?

A P, Q, R B Q, P, R C R, Q, P D R, P, Q E P, R, Q

Solution **A**

We choose units so that the two equal sides of the triangles that make up the shapes have length 1. It then follows, by Pythagoras' Theorem, that the length of the hypotenuse of each of these triangles is $\sqrt{2}$.



We let p, q and r be the lengths of the perimeters of the shapes P, Q and R, respectively.

From the diagrams given in the question we see that $p = 2 + 3\sqrt{2}$, $q = 6 + \sqrt{2}$, and $r = 4 + 3\sqrt{2}$.

Commentary

To complete the answer to this question it helps to start with a guess based on the rough approximation $\sqrt{2} \approx 1.5$. This gives $p \approx 2 + 4.5 = 6.5$, $q \approx 6 + 1.5 = 7.5$, and $r \approx 4 + 4.5 = 8.5$.

This suggests that $p < q < r$. We check this using two methods guided by our guess.

The first method uses the fact that $1.4 < \sqrt{2} < 1.5$. The second uses the more elementary fact that, because $1 < 2 < 4$, we have $1 < \sqrt{2} < 2$.

Method 1

Since $1.4 < \sqrt{2} < 1.5$, we have

$$2 + 3\sqrt{2} < 2 + 3 \times 1.5 = 6.5,$$

$$7.4 = 6 + 1.4 < 6 + \sqrt{2} < 6 + 1.5 = 7.5,$$

$$\text{and} \quad 8.2 = 4 + 3 \times 1.4 < 4 + 3\sqrt{2}.$$

Therefore we have

$$p < 6.5 < 7.4 < q < 7.5 < 8.2 < r.$$

From this we see that $p < q < r$ and hence the shapes in ascending order of the lengths of their perimeters are P, Q, R.

METHOD 2

We have

$$\begin{aligned} q - p &= (6 + \sqrt{2}) - (2 + 3\sqrt{2}) \\ &= 4 - 2\sqrt{2} \\ &= 2(2 - \sqrt{2}). \end{aligned}$$

Since $\sqrt{2} < 2$, it follows that $2 - \sqrt{2} > 0$. Hence $q - p > 0$ and so $p < q$.

Similarly,

$$\begin{aligned} r - q &= (4 + 3\sqrt{2}) - (6 + \sqrt{2}) \\ &= 2\sqrt{2} - 2 \\ &= 2(\sqrt{2} - 1). \end{aligned}$$

Since $1 < \sqrt{2}$, it follows that $\sqrt{2} - 1 > 0$. Hence $r - q > 0$ and so $q < r$.

Therefore $p < q < r$ and the shapes in ascending order of their perimeters are P, Q, R.

FOR INVESTIGATION

23.1 Calculate the values of 1.4^2 and 1.5^2 and hence verify that $1.4 < \sqrt{2} < 1.5$.

23.2 Which is the larger $3 + 15\sqrt{2}$ or $20 + 3\sqrt{2}$? (Do not use a calculator!)

24. The positive integers m and n are such that $10 \times 2^m = 2^n + 2^{n+2}$.
What is the difference between m and n?

A 1 B 2 C 3 D 4 E 5

SOLUTION **A**

We have $10 \times 2^m = 5 \times 2 \times 2^m = 5 \times 2^{m+1}$ and $2^n + 2^{n+2} = 2^n(1 + 2^2) = 2^n \times 5$.

Therefore the equation $10 \times 2^m = 2^n + 2^{n+2}$ may be rewritten as $5 \times 2^{m+1} = 2^n \times 5$. It follows that $2^{m+1} = 2^n$ and hence $m + 1 = n$.

We deduce that the difference between m and n is 1.

FOR INVESTIGATION

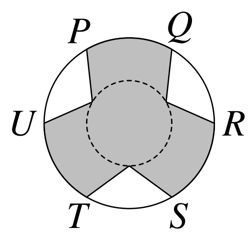
24.1 Find all the pairs of positive integers m and n that are solutions of the equation

$$3^m \times 2^n = 2^n + 2^{n+1} + 2^{n+3} + 2^{n+4}.$$

24.2 Find all the positive integers n that are solutions of the equation

$$3^n + 5^{n+2} = 2^{n+6}.$$

25. The diagram shows six points P, Q, R, S, T and U equally spaced around a circle of radius 2 cm. The inner circle has radius 1 cm. The shaded region has three lines of symmetry. Which of the following gives the area, in cm^2, of the shaded region?

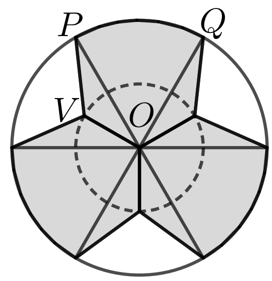


A $2\pi + 3$ B $3\pi + 2$ C $\frac{4\pi + 3}{2}$

D $3(\pi + 2)$ E $4\pi + 3$

Solution **A**

Let O be the common centre of the two circles, and let V be the point shown.



The shaded region is made up of three sectors of the circle and six triangles.

Because of the symmetry of the shaded region, the three sectors are all congruent to the sector POQ, and the six triangles are all congruent to the triangle POV.

We can also deduce from the symmetry of the figure that $\angle POQ = 60°$.

It follows that the area of each sector is one-sixth of the area of the circle with radius 2 cm. Therefore the total area of these three sectors is half the area of this circle.

Hence the total area of the three sectors is given by

$$\tfrac{1}{2} \times (\pi \times 2^2)\,\text{cm}^2 = 2\pi\,\text{cm}^2.$$

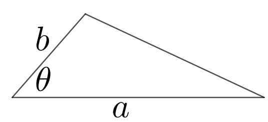
From the symmetry of the figure we can also deduce that $\angle POV = 30°$.

Therefore, in the triangle POV, we have $OV = 1$ cm, $OP = 2$ cm and $\angle POV = 30°$. Hence the area of this triangle is given by $\frac{1}{2}OV.OP.\sin(30°)\,\text{cm}^2 = (\frac{1}{2} \times 1 \times 2 \times \frac{1}{2})\,\text{cm}^2 = \frac{1}{2}\,\text{cm}^2$. Therefore the total area of the six triangles is $6 \times \frac{1}{2}\,\text{cm}^2 = 3\,\text{cm}^2$.

It follows that the area of the shaded region, in cm^2, is $2\pi + 3$.

For investigation

25.1 To find the area of the triangle POV we used the formula



$$\tfrac{1}{2}ab\sin\theta$$

for the area of a triangle which has sides of lengths a and b and with θ as the included angle.

Prove that this formula for the area is correct.

25.2 What is the area of the unshaded region inside the circle of radius 2 cm?