Junior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

30 April 2019

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are often short so that they all fit on one sheet of paper. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, occasionally, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	E	A	E	C	B	A	A	B	E	D	D	E	D	E	C	D	C	B	E	D	A	B	B	D

1. How many minutes is it from 23:35 today to 01:15 tomorrow?

A 100 B 110 C 120 D 130 E 140

SOLUTION **A**

There are 25 minutes from 23.35 until midnight, and 75 minutes from midnight to 01:15 on the following day.

Therefore the number of minutes from 23:35 to 01:15 on the following day is 25 + 75 = 100.

FOR INVESTIGATION

1.1 How many minutes is it from 01.15 today to 23:35 tomorrow?

1.2 What time is it 1000 minutes after 23:35 today?

2. Which of these is equal to $(0.1 + 0.2 + 0.3 - 0.4) \div 0.5$?

A 0.01 B 0.02 C 0.04 D 0.1 E 0.4

SOLUTION **E**

We have

$$0.1 + 0.2 + 0.3 = 0.6$$

and therefore

$$0.1 + 0.2 + 0.3 - 0.4 = 0.6 - 0.4 = 0.2.$$

It follows that

$$(0.1 + 0.2 + 0.3 - 0.4) \div 0.5 = 0.2 \div 0.5.$$

Now

$$0.2 \div 0.5 = \frac{0.2}{0.5} = \frac{2}{5} = 0.4.$$

Therefore

$$(0.1 + 0.2 + 0.3 - 0.4) \div 0.5 = 0.4.$$

FOR INVESTIGATION

2.1 Which of these expressions has the largest value?

(a) $0.1 + 0.2 + 0.3 + 0.4 + 0.5$

(b) $0.1 + 0.2 + 0.3 + 0.4 \div 0.5$

(c) $0.1 \div 0.2 + 0.3 + 0.4 + 0.5$

(d) $0.1 \div (0.2 \times 0.3) + 0.4 + 0.5$

3. Sam has eaten three-quarters of the grapes.

What is the ratio of the number of grapes that remain to the number Sam has eaten?

A 1 : 3   B 1 : 4   C 1 : 5   D 1 : 6   E 1 : 7

SOLUTION **A**

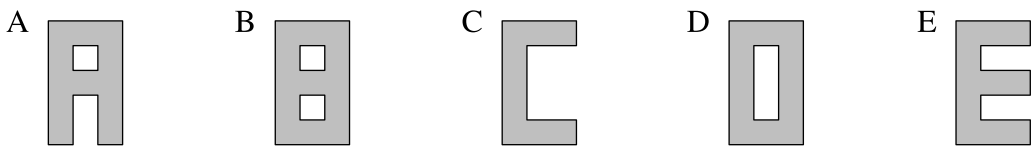
When Sam has eaten three-quarters of the grapes, one quarter of the grapes are left. Therefore the ratio of the number of grapes left to the number Sam has eaten is

$$\tfrac{1}{4} : \tfrac{3}{4}.$$

This is the same as

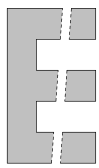
$$1 : 3.$$

4. Which of the following five shapes can be cut into four pieces by a single straight cut?



SOLUTION **E**

It can be seen that if the shape of option E is cut as shown in the diagram, it is cut into four pieces by a single straight cut.



In the context of the JMC you can assume that just one of the options is correct. Therefore, having found that the shape of option E can be cut into four pieces by a single straight cut, there is no need to consider the other shapes.

This is fortunate because, although it is not difficult to 'see' that none of the other shapes can be cut into four pieces by a single straight cut, writing an argument in words to prove this would be rather hard.

FOR INVESTIGATION

4.1 For each of the other shapes, find the maximum number of pieces into which it can be cut by a single straight cut.

4.2 What is the maximum number of pieces into which the shape shown on the right can be cut by a single straight cut?

4.3 Draw a shape that can be cut into exactly 10 pieces by a single straight cut.

5. On Aoife's 16th birthday, Buster was three times her age. On Aoife's 21st birthday, how old was Buster?

A 32 B 48 C 53 D 63 E 64

SOLUTION **C**

When Aoife age was 16, Buster's age was $3 \times 16 = 48$. Therefore Buster is $48 - 16 = 32$ years older than Aoife.

It follows that when Aoife was 21, Buster's age was $21 + 32 = 53$.

FOR INVESTIGATION

5.1 On Mollie's 40th birthday, Daniel was one fifth of Mollie's age. How old was Daniel on Mollie's 50th birthday?

6. Which of these is closest to 7?

A 7.09 B 6.918 C 7.17 D 6.7 E 7.085

SOLUTION **B**

The option that is closest to 7 is the one who difference from 7 is smaller than for any of the other options.

We have

$$7.09 - 7 = 0.09,$$
$$7 - 6.918 = 0.082,$$
$$7.17 - 7 = 0.17,$$
$$7 - 6.7 = 0.3$$

and

$$7.085 - 7 = 0.085.$$

We see that the smallest of these differences is 0.082. It follows that, of the given numbers, it is 6.918 that is closest to 7.

FOR INVESTIGATION

6.1 Which of these is closest to 7?

(a) 7.1, (b) 7.01, (c) 7.001, (d) 7.0001, (e) 7.00001.

6.2 Is there a number x such that x not equal to 7 and x is closer to 7 than any other number which is not equal to 7?

7. The shortest street in the UK, *Ebenezer Place* in Wick, is 2.06 m long. The *Trans-Canada Highway*, one of the world's longest roads, is approximately 7821 km in length.

Approximately, how many times longer than the street is the highway?

A 4 000 000 B 400 000 C 40 000 D 4000 E 400

SOLUTION **A**

Since 1 km = 1000 m, it follows that 7821 km = 7821 × 1000 m. Therefore

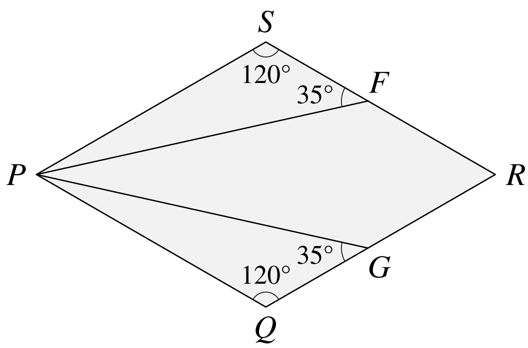
$$\frac{7821\text{ km}}{2.06\text{ m}} = \frac{7821 \times 1000}{2.06} \approx \frac{8000 \times 1000}{2} = 4000 \times 1000 = 4\,000\,000.$$

Therefore the highway is approximately 4 000 000 times longer than the street.

8. The diagram shows a kite $PGRF$ inside rhombus $PQRS$. Angle $PGQ = 35°$, angle $PFS = 35°$, angle $PQG = 120°$ and angle $PSF = 120°$.

What is the size of angle FPG?

A 10° B 12° C 15° D 18° E 20°



SOLUTION **A**

The sum of the angles in a triangle is 180°. Therefore, in the triangle PFS we have

$$\angle FPS + 35° + 120° = 180°,$$

and therefore

$$\angle FPS = 180° - 35° - 120° = 25°.$$

Similarly, $\angle QPG = 25°$.

Because $PGRF$ is a rhombus, SR is parallel to PQ. Hence

$$\angle QPS + \angle PSF = 180°.$$

Therefore

$$\angle QPS = 180° - 120° = 60°.$$

We can now deduce that

$$\angle FPG = \angle QPS - \angle FPS - \angle QPG = 60° - 25° - 25° = 10°.$$

9. What is 50% of 18.3 plus 18.3% of 50?

A 9.15 B 18.3 C 27.15 D 59.15 E 68.3

SOLUTION **B**

METHOD 1

We have

$$\begin{aligned} 50\% \text{ of } 18.3 \text{ plus } 18.3\% \text{ of } 50 &= \frac{50}{100} \times 18.3 + \frac{18.3}{100} \times 50 \\ &= \frac{50 \times 18.3}{100} + \frac{18.3 \times 50}{100} \\ &= 2 \times \left(\frac{50 \times 18.3}{100}\right) \\ &= \frac{100 \times 18.3}{100} \\ &= 18.3. \end{aligned}$$

METHOD 2

In the context of the JMC where we have just five options to choose from, we can avoid the need to do an exact calculation. Instead, a rough approximation leads us to the correct option.

We know that 50% is the same as a half. A half of 18.3 is approximately 9.

Also 18.3% is approximately 20% which is a fifth. A fifth of 50 is 10.

Therefore 50% of 18.3 plus 18.3% of 50 is approximately 9 + 10 = 19.

We can safely conclude that that the correct option is B.

FOR INVESTIGATION

9.1 What is 40% of 12.5 + 12.5% of 40?

9.2 Show that, in general, a% of b is equal to b% of a.

10. What is the last digit of the smallest positive integer whose digits add to 2019?

A 1 B 4 C 6 D 8 E 9

Solution **E**

Given two positive integers made up of different numbers of digits, the integer which has the smaller number of digits will be the smaller of the two integers. [For example, 9 999 999 which has seven digits is smaller than 10 000 000 which has eight digits.]

Therefore to obtain the smallest postive integer whose digits add up to 2019, we seek positive integers with the smallest possible number of digits whose digits add up to 2019. These will be the positive integers which contains the largest possible number of copies of the digit 9.

Now $2019 \div 9 = 224$ with remainder 3. Therefore the smallest positive integer whose digits add up to 2019 will have 225 digits, of which 224 are 9s, with the other digit being 3.

The smallest such positive integer consists of a single 3 followed by 224 copies of 9. That is, it is the number

$$3\overbrace{999\ldots999}^{224}.$$

The last digit of this positive integer is 9.

For investigation

10.1 Note that to solve this problem it wasn't necessary to calculate $2019 \div 9$, as for each positive integer n with $n \geq 9$, the last digit of the smallest integer whose digits add up to n is 9.

Prove that this statement is correct.

10.2 What is the largest positive integer which does not use the digit 0 and whose digits have sum 2019?

10.3 Let M be the smallest positive integer whose digits add up to 2019, and let N be the smallest positive integer whose digits add up to 2020.

What is the value of $N - M$?

10.4 What is the remainder when the number

$$3\overbrace{999\ldots999}^{224}.$$

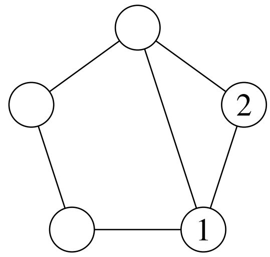
is divided by 9?

10.5 What is the remainder when the number

$$3\overbrace{999\ldots999}^{224}.$$

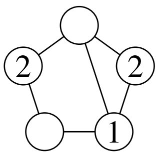
is divided by 7?

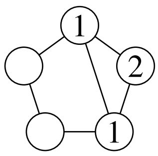
11. Two players X and Y take alternate turns in a game, starting with the diagram alongside.

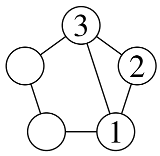


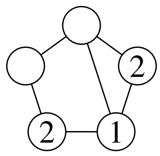
At each turn, one player writes one of 1, 2 or 3 in an empty circle, so that no two circles connected by an edge contain the same number. A player loses when they cannot go. In each of the five diagrams below it is Y's turn.

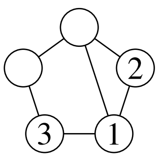
In which of the diagrams can Y's move ensure that X loses the game?

A


B


C


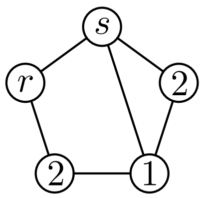
D


E


Solution **D**

Note: Position B was included by error, as it cannot occur in the game, and is ignored in what follows.

The figure on the right shows the diagram of option D in which we have labelled the circles r and s, as shown, so that we may refer to them.



Suppose that player Y now writes 3 in the circle labelled r. The only remaining empty circle is the circle labelled s. It is now connected by edges to circles containing 1, 2 and 3. Therefore there is no legitimate move for player X. So player X loses. Therefore in position D player Y can ensure that X loses.

In the context of the JMC it is good enough to identify one of the options which is a definite loss for X. For a complete solution it is necessary to check that in none of the other positions can player Y ensure a win. This is left as an exercise for the reader.

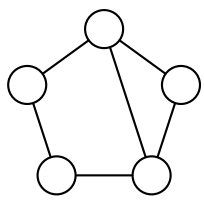
For investigation

11.1 Show that if in position D player Y makes any move other than writing 3 in the circle labelled r, then player X wins.

11.2 Show that player X wins each of the positions of options A, C and E.

11.3 In this game player X makes the first move in the initial position shown on the right.

Which player can force a win?



12. Jamal writes down a sequence of six integers. The rule he uses is, "after the first three terms, each term is the sum of the three previous terms." His sequence is —, —, —, 8, 13, 25.

What is his first term?

A 0 B 1 C 2 D 3 E 4

SOLUTION **D**

METHOD 1

We work backwards.

The sixth term of the sequence is the sum of the previous three terms. Therefore 25 is the sum of 13, 8 and the third term. Hence the third term is 4.

Similarly, the fifth term is the sum of the previous three terms. Therefore 13 is the sum of 8, 4 and the second term. Hence the second term is 1.

Similarly, the fourth term, 8, is the sum of 4, 1 and the first term. Therefore the first term is 3.

METHOD 2

[This is the previous method expressed in terms of algebra.]

Let the first three terms of the sequence be x, y and z. Therefore the sequence is

$$x,\ y,\ z,\ 8,\ 13,\ 25.$$

Because each term after the first three is the sum of the three previous terms, we have

$$\begin{aligned} x + y + z &= 8, &(1)\\ y + z + 8 &= 13, &(2)\\ z + 8 + 13 &= 25. &(3) \end{aligned}$$

From equation (3) it follows that $z = 25 - 8 - 13 = 4$.

Therefore, from equation (2), $y = 13 - 8 - z = 13 - 8 - 4 = 1$.

Hence, from equation (1), $x = 8 - y - z = 8 - 1 - 4 = 3$.

Therefore Jamal's first term is 3.

FOR INVESTIGATION

12.1 In the sequence

—, —, —, —, —, 281, 550, 1044, 2019

each term after the first four is the sum of the previous four terms.

Find the first term of the sequence.

13. In how many different ways can you spell out JMC, starting at the centre, and moving to the next letter in a neighbouring square – horizontally, vertically, or diagonally – each time?

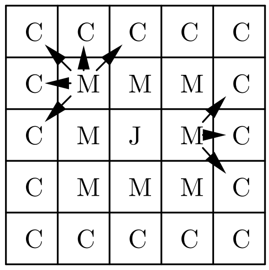
A 8 B 16 C 24 D 25 E 32

C	C	C	C	C
C	M	M	M	C
C	M	J	M	C
C	M	M	M	C
C	C	C	C	C

Solution **E**

Method 1

Each of the squares labelled M can be reached in just one way from the central square which is labelled J.



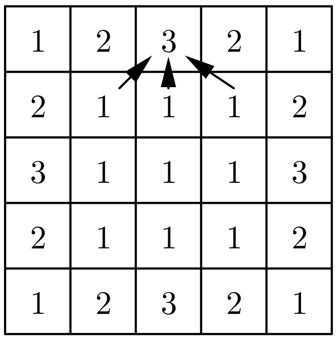
Among the eight squares labelled M, there are four squares from which there are five squares labelled C that can be reached. One of these is shown in the figure on the right. This gives, altogether, $4 \times 5 = 20$ ways to spell out JMC.

From each of the other four squares labelled M there are three squares labelled C that can be reached. One of these is shown in the figure. This gives another $4 \times 3 = 12$ ways to spell out JMC.

Therefore the total number of ways in which it is possible to spell out JMC is $20 + 12 = 32$.

Method 2

Another way to view the problem is that the number of ways to spell out JMC is the same as the number of ways to get from the centre square to an edge square in exactly two moves.



We can solve this problem by counting the number of ways to reach each square from the centre square in the minimum possible number of moves. We have inserted these numbers in the diagram on the right, and now explain how they are calculated.

Each square adjacent to the centre square can be reached in one move in just one way. So we have put 1 in all these squares.

An edge square, say X, can be reached in two moves or more. The number of ways to reach X in just two moves is the sum of the numbers in the squares already numbered which are one move away from X.

For example, the square in the middle of the top row may be reached in one move from each of the three squares below it, as indicated by the arrows.

Hence we can reach this square in $1 + 1 + 1$, that is, 3, ways.

Finally, we add up all the numbers in the edge squares. It can be checked that this sum is 32. Hence we can reach an edge square in exactly two moves from the centre square in 32 ways.

We deduce that we can spell out JMC in 32 ways.

FOR INVESTIGATION

13.1 Check that in method 2 the numbers in the edge squares are all correct, and that their sum is 32.

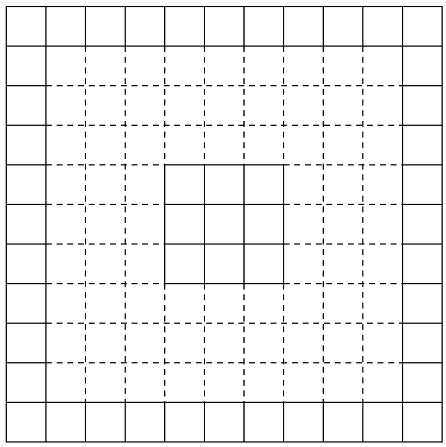
13.2 In how many ways can you spell out UKMT, starting at the central U, and moving to the next letter in a neighbouring square - horizontally, vertically, or diagonally - each time?

T	T	T	T	T	T	T
T	M	M	M	M	M	T
T	M	K	K	K	M	T
T	M	K	U	K	M	T
T	M	K	K	K	M	T
T	M	M	M	M	M	T
T	T	T	T	T	T	T

13.3 Consider a $2n + 1$ by $2n + 1$ grid of squares.

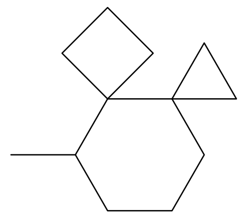
Find a formula,in terms of n, for the total number of different ways of reaching an edge square from the centre square in n moves.

As before a move consists of moving to a neighbouring square either horizontally, vertically or diagonally.



14. Each edge in the diagram has length 1 cm.

What is the length of the longest path that can be followed along the edges, starting at a vertex and without revisiting any vertex?



A 7 cm B 8 cm C 9 cm D 10 cm E 11 cm

Solution **D**

In the figures in this solution the vertices have been labelled so that we can refer to them.

A path which does not revisit a vertex can use at most five edges of the hexagon, three edges of the square, two edges of the triangle and the edge SZ.

However, as we now show, a path which uses the edge SZ together with five edges of the hexagon and three edges of the square, without revisiting a vertex, cannot use any of the edges of the triangle.

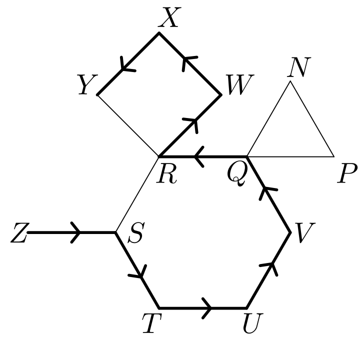


Figure 1

A path of with these properties must go through the vertices R and S. As it does not revisit any vertex, it cannot use the edge RS, but instead uses the other five edges of the hexagon.

It follows that such a path either uses the three edges RW, WX and XY of the square, or the three edges RY, YX and XW.

We therefore see that there are only four paths that have the required properties. These are the two paths shown in Figures 1 and 2, and the paths using the same edges, but taken in reverse order.

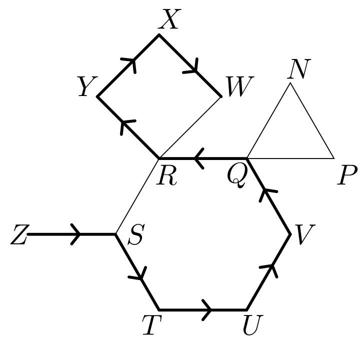


Figure 2

None of these four paths includes any of the edges of the triangle.

Therefore the longest path which does not revisit a vertex can use at most five edges of the hexagon, three edges of the square and two edges of the triangle, but not the edge SZ. Such a path has at most $5 + 3 + 2 = 10$ edges. The path in Figure 3 shows that such a path with 10 edges exists.

Each edge has length 1 cm. Hence the maximum length of a path that does not revisit a vertex is 10 cm.

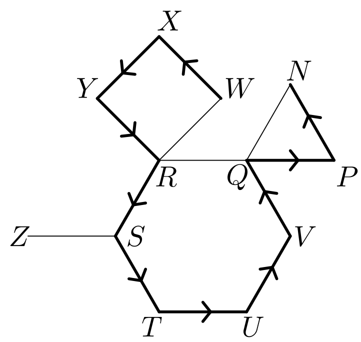


Figure 3

For investigation

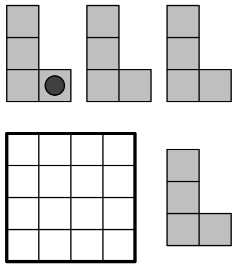
14.1 How many different paths of length 10 cm are there that do not revisit a vertex?

14.2 What is the length of the longest path that can be followed along the edges, starting at a vertex, and not using any edge more than once (but possibly revisiting vertices)?

15. All four L-shapes shown in the diagram are to be placed in the 4 by 4 grid so that all sixteen cells are covered and there is no overlap. Each piece can be rotated or reflected before being placed and the black dot is visible from both sides.

How many of the 16 cells of the grid could contain the black dot?

A 4 B 7 C 8 D 12 E 16



Solution **E**

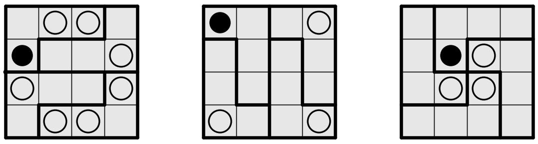


Figure 1 Figure 2 Figure 3

The figures show three arrangements of the four L-shapes.

The solid circle shows the position of the black dot.

The hollow circles show the position that the black dot could take if we rotate or reflect the grid.

For example, by reflecting Figure 1 about the horizontal line through the middle of the grid, we see that the black dot could be in the first cell from the left in the third row from the top.

We note that the solid and hollow circles between them occupy all 16 cells of the grid. Therefore each of the 16 cells could contain the black dot.

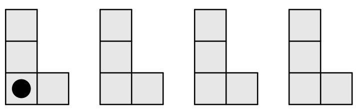
For investigation

15.1 Check that in each of the three figures above, the hollow circles show the positions that the black dot could take by rotating or reflecting the figure.

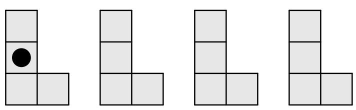
15.2 Each of the following set of four L-shapes is used to cover all 16 cells of a 4 by 4 grid without any overlaps. Each L-shape can be rotated or reflected before being placed and the black dot is visible from both sides.

In each case, how many of the 16 cells of the grid could contain the black dot?

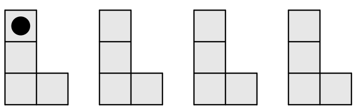
(a)



(b)



(c)



16. Tamsin writes down three two-digit integers. One is square, one is prime and one is triangular.
She uses the digits 3, 4, 5, 6, 7 and 8 exactly once each.

Which prime does she write?

A 37 B 43 C 53 D 73 E 83

Solution **C**

Commentary

It is not immediately clear where is the best place to begin tackling this problem.

There are lots of two digit primes that use just the given digits, but in the context of the JMC, it would only be necessary to consider those given as options, giving five cases to consider.

However, as there are only two two-digit squares which use just the digits 3, 4, 5, 6, 7 and 8, it is best to begin by considering the squares.

The two-digit squares are

$$16,\ 25,\ 36,\ 49,\ 64,\ 81.$$

It can be seen that only two of these, 36 and 64, use just the given digits. We consider these in turn.

If Tamsin writes 36 as her square, she is left with the digits 4, 5, 7 and 8.

The two-digit triangular numbers are

$$10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91.$$

Of these only 45 and 78 use two digits from 4, 5, 7 and 8.

If Tamsin now chooses 45 as her triangular number, she is left with the digits 7 and 8. However, neither 78 nor 87 is a prime, so this choice doesn't work.

If Tamsin chooses 78 as her triangular number, she is left with the digits 4 and 5. This also doesn't work as neither 45 nor 54 is a prime.

We see from this that Tamsin needs to choose 64 as her square. This leaves the digits 3, 5, 7 and 8 with which to make a triangular number and a prime.

Using two of these, the only triangular number Tamsin can make is 78. This leaves her with the digits 3 and 5. Now 35 is not prime, but 53 is.

We conclude that the prime that Tamsin writes is 53.

For investigation

16.1 List all the two-digit primes that use two of the digits 3, 4, 5, 6, 7 and 8.

16.2 List all the two-digit triangular numbers that use two of the digits 3, 4, 5, 6, 7 and 8.

17. A rectangle is three times as long as it is high. The area of a square is twelve times the area of the rectangle. What is the ratio of the perimeter of the square to the perimeter of the rectangle?

A 12 : 1 B 6 : 1 C 4 : 1 D 3 : 1 E 2 : 1

SOLUTION **D**

Let S be the perimeter of the square, and let R be the perimeter of the rectangle. We need to find the ratio $S : R$.

Let the height of the rectangle be h. It follows that the length of the rectangle is $3h$. Hence the area of the rectangle is $h \times 3h = 3h^2$.

The rectangle has two sides of length h and two sides of length $3h$. The perimeter of the rectangle is the sum of these side lengths. Hence

$$R = h + 3h + h + 3h = 8h.$$

Let the side length of the square be s. Then the area of the square is s^2.

Also, the perimeter of the square is given by

$$S = s + s + s + s = 4s.$$

Because the area of the square is twelve times the area of the rectangle,

$$s^2 = 12 \times 3h^2 = 36h^2.$$

Hence, as h and s are positive, $s = 6h$.

It follows that

$$\frac{S}{R} = \frac{4s}{8h} = \frac{4 \times 6h}{8h} = \frac{24h}{8h} = 3.$$

Therefore

$$S : R = 3 : 1.$$

FOR INVESTIGATION

17.1 A rectangle is four times as long as it is high. The area of a square is nine times the area of the rectangle.

What is the ratio of the perimeter of the square to the perimeter of the rectangle?

18. What fraction of the integers from 1 to 8000 inclusive are cubes?

A $\frac{1}{1000}$ B $\frac{1}{800}$ C $\frac{1}{400}$ D $\frac{1}{200}$ E $\frac{1}{100}$

Solution **C**

We have $1^3 = 1$ and $20^3 = 8000$. Therefore the cubes from 1 to 8000 inclusive are the cubes of the first 20 positive integers. Hence there are 20 cubes from 1 to 8000 inclusive.

Because there are 8000 integers from 1 to 8000, it follows that the fraction of these integers that are cubes is

$$\frac{20}{8000}$$

which is equal to

$$\frac{1}{400}.$$

19. Each row, each column and each of the bold 2 by 3 rectangles in the grid has to contain each of the numbers 1, 2, 3, 4, 5 and 6 (one number in each cell).

What number should go in the cell marked x?

A 1 B 2 C 3 D 4 E 6

				x	5
				6	
		1	2		
		3	4		
		4		3	
2					1

Solution **B**

In the figure on the right the three other vacant cells in the fifth column have been marked r, s and t so that we can refer to them.

				x	5
				6	
		1	2	r	
		3	4	s	
		4		3	
2				t	1

The number 2 must be in one of the vacant cells in the fifth column.

The 2 cannot be in either of the cells marked r and s because there is already a 2 in the bold 2 by 3 rectangle in which they are situated.

The 2 cannot be in the cell marked t because there is already a 2 in its row.

We deduce that the number 2 is in the cell marked x.

Note: This answer assumes, as does the question, that it is possible to fill all the empty cells in the grid in the way described. You are asked to check this in Exercise 19.1.

For investigation

19.1 In how many different ways can the empty cells in the grid given in the question be filled so that each row, each column and each bold 2 by 3 rectangle contains the numbers 1, 2, 3, 4, 5 and 6?

> **20.** Emily writes down the largest two-digit prime such that each of its digits is prime.
> Krish writes down the smallest two-digit prime such that each of its digits is prime.
> Kirsten subtracts Krish's number from Emily's number.
> What answer does Kirsten obtain?
>
> A 14 B 20 C 36 D 45 E 50

SOLUTION **E**

The digits that are primes are 2, 3, 5 and 7. A two-digit integer whose units digit is 2 is divisible by 2, and hence it not prime. A two-digit integer whose units digit is 5 is divisible by 5 and hence is not prime. Hence the only possible primes that can be the units digit of a two-digit prime are 3 and 7.

The integers 33 and 77 are divisible by 11, and hence are not prime.

The integers 27 and 57 are divisible by 3 and hence are not prime.

Hence, the only remaining possibilities for a two-digit number both of whose digits are prime, and which is itself a prime are

$$23, 37, 53 \text{ and } 73.$$

It can be checked that these are all primes.

It follows that the largest such prime is 73 and the smallest is 23. Therefore Emily writes down 73 and Krish writes down 23. Hence Kirsten's subtraction sum is $73 - 23$. Therefore the answer that Kirsten obtains is 50.

> It is important to remember that 1 is not regarded as a prime, so that 11 is *not* the smallest two-digit prime both of whose digits are prime.
>
> It is only a convention that 1 does not count as a prime, but it is a standard convention.
>
> As Vicky Neale says in her recent book, *Closing the Gap: the quest to understand prime numbers*, Oxford University Press, 2017,
>
> "In case you're wondering, I'll mention that 1 is not prime. That's not because of some interesting philosophical point, or because it's only got one number that divides into it rather than two, or anything like that. Definitions (such as that of a prime number) don't get handed to mathematicians on stone tablets. Rather, part of the job of mathematicians is to make good definitions, ones that lead to interesting mathematics, and it turns out to be better to define 1 not to be prime."

FOR INVESTIGATION

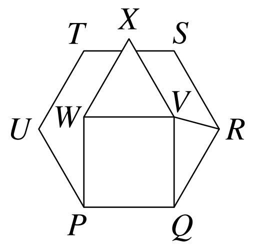
20.1 Check that each of the numbers 23, 37, 53 and 73 is prime.

20.2 Find all the three-digit primes each of whose digits is a prime.

21. The diagram shows a regular hexagon $PQRSTU$, a square $PQVW$ and an equilateral triangle VXW.

What is the size of angle XVR?

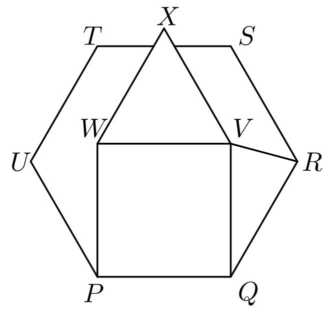
A 120° B 125° C 130° D 135° E 140°



Solution **D**

Because VXW is an equilateral triangle,

$$\angle XVW = 60°. \qquad (1)$$



Because $PQVW$ is a square,

$$\angle QVW = 90°. \qquad (2)$$

Because $PQRSTU$ is a regular hexagon and $PQVW$ is a square,

$$\angle RQV = \angle RQP - \angle VQP = 120° - 90° = 30°.$$

Because $PQVW$ is a square and $PQRSTU$ is a regular hexagon, $VQ = QP = RQ$. Therefore VQR is an isosceles triangle. Hence $\angle QVR = \angle QRV$.

The angles of a triangle add up to 180°. Therefore from the triangle VQR we have $\angle RQV + \angle QVR + \angle QRV = 180°$. Therefore $30° + 2\angle QVR = 180°$, and hence

$$\angle QVR = \tfrac{1}{2}(180° - 30)° = 75°. \qquad (3)$$

The sum of the angles at a point is 360°. Therefore, from the angles at the point V we have

$$\angle XVR + \angle XVW + \angle QVW + \angle QVR = 360°.$$

Hence, by (1), (2) and (3), it follows that

$$\angle XVR = 360° - 60° - 90° - 75° = 135°.$$

For investigation

21.1 What is the size of the angle XSR?

21.2 Find a proof that the sum of the angles in a triangle is 180°. [That is, find your own proof, or look in a book or on the web, or ask your teacher.]

21.3 (a) Find a formula in terms of n for the size, in degrees, of the interior angles of a regular polygon with n sides.

(b) Deduce that the interior angle of a regular hexagon is 120°.

22. In the multiplication shown alongside, T, R, A and P are all different digits.

What is the value of R?

$$\begin{array}{cccc} T & R & A & P \\ \times & & & 9 \\ \hline P & A & R & T \\ \hline \end{array}$$

A 0 B 1 C 5 D 8 E 9

SOLUTION **A**

Both '$TRAP$' and '$PART$' are four-digit integers and so are in the range from 1000 to 9999. Since '$TRAP$' $\times 9 =$ '$PART$' $< 10\,000$, it follows that

$$\text{'}TRAP\text{'} < \frac{10\,000}{9} < 1112.$$

The first digit of '$TRAP$' cannot be 0. Therefore, $T = 1$, and, as $R \neq T$, we deduce that $R = 0$.

FOR INVESTIGATION

22.1 Find the values of A and P.

22.2 There is one more example of a four-digit integer '$TRAP$' and a digit M with $M > 0$ such that the following sum is correct.

$$\begin{array}{cccc} T & R & A & P \\ \times & & & M \\ \hline P & A & R & T \\ \hline \end{array}$$

Find the values of '$TRAP$' and M, with $M \neq 9$.

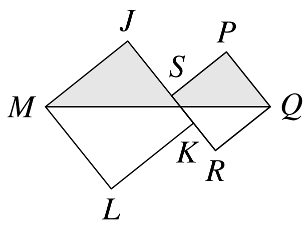
22.3 '$DENIM$' is a five-digit integer, whose digits are all different, and Y is a digit such that the following sum is correct.

$$\begin{array}{ccccc} D & E & N & I & M \\ \times & & & & Y \\ \hline M & I & N & E & D \\ \hline \end{array}$$

Find the values of '$DENIM$' and Y.

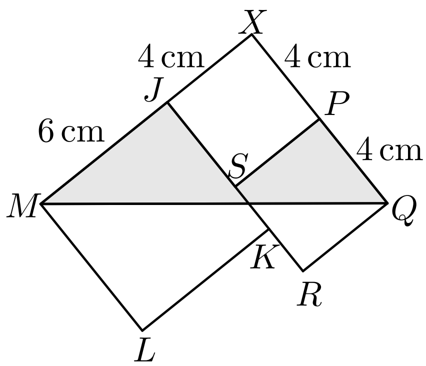
23. The diagram shows two squares $JKLM$ and $PQRS$.
The length of JK is 6 cm and that of PQ is 4 cm.
The vertex K is the midpoint of side RS.
What is the area of the shaded region?

A $22\,\text{cm}^2$ B $24\,\text{cm}^2$ C $26\,\text{cm}^2$
D $28\,\text{cm}^2$ E $30\,\text{cm}^2$



SOLUTION **B**

We let X be the point where the line MJ when extended meets the line QP when extended, as shown in the diagram.



Because K is the midpoint of RS, we have $SK = 2\,\text{cm}$. Hence $SJ = KJ - KS = 6\,\text{cm} - 2\,\text{cm} = 4\,\text{cm}$. Hence $SJ = SP$.

Because $JKLM$ and $PQRS$ are both squares, $\angle JSP = \angle SJX = \angle SPX = 90°$. It follows that $SPXJ$ is a square with side length 4 cm. Hence the area of $SPXJ$ is $16\,\text{cm}^2$.

It also follows that $QX = QP + PX = 8\,\text{cm}$ and $MX = MJ + JX = 10\,\text{cm}$. Therefore, because the triangle XMQ has a right angle at X, the area of this triangle is $\frac{1}{2}(8 \times 10)\,\text{cm}^2$, that is, $40\,\text{cm}^2$.

The shaded area is the area of the triangle XMQ less the area of the square $SPXJ$. Therefore the shaded area is $40\,\text{cm}^2 - 16\,\text{cm}^2 = 24\,\text{cm}^2$.

FOR INVESTIGATION

23.1 Let T be the point where MQ meets SR.

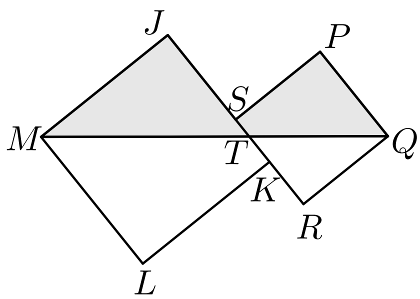
(a) Find the length of ST.

(b) Calculate the area of the triangle MTJ.

(c) Calculate the area of the trapezium $STQP$.

(d) Hence find the area of the shaded region.

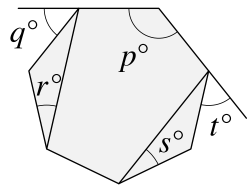
[Hint for (a): look for a pair of similar triangles.]



24. The diagram shows a regular heptagon.

Which of these expressions is equal to $p + q + r + s + t$?

A $180 + q$ B $180 + 2q$ C $360 - q$
D 360 E $360 + q$



SOLUTION **B**

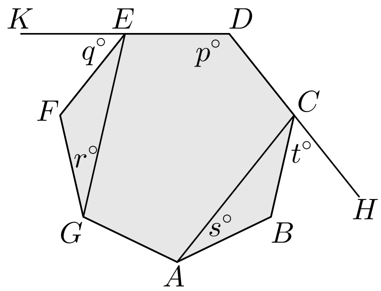
COMMENTARY

Because we are dealing with a regular heptagon, we could work out the numerical values of p, q, r, s and t. We could then do a calculation to show that $p+q+r+s+t = 180+2q$. We leave this method to Exercise 24.1.

Instead, we give a method which enables us to show that option B is correct without the need to evaluate the individual angles.

The question suggests this approach because the answer generalizes to other regular polygons. We ask you to follow this idea in Exercise 24.2

We label the diagram as shown.



Because $ABCDEFG$ is a regular heptagon, $AB = BC = EF = FG$ and $\angle CBA = \angle GFE$. Therefore ABC and EFG are congruent isosceles triangles. It follows that $\angle GEF = \angle ACB = \angle CAB = s°$.

Also, because the heptagon is regular, $\angle GFE = \angle CDE = p°$.

The angles of a triangle have sum 180°. Therefore from the triangle EFG, we have $\angle EFG + \angle EGF + \angle GEF = 180°$. Therefore $p + r + s = 180$.

Because $\angle KEF$ and $\angle HCB$ are both exterior angles of the regular heptagon, they are equal. Therefore $t = q$.

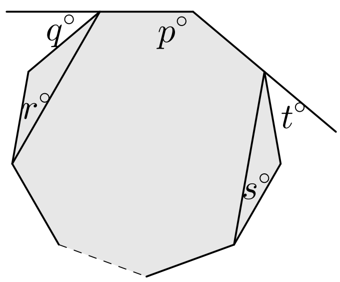
It follows that $p + q + r + s + t = (p + r + s) + q + t = 180 + q + q = 180 + 2q$.

FOR INVESTIGATION

24.1 Find the numerical values of p, q, r, s and t, and hence show that $p+q+r+s+t = 180+2q$.

24.2 The diagram shows a regular polygon with n sides, where $n \geq 6$.

Show that $p + q + r + s + t = 180 + 2q$.



25. The diagram shows the first fifteen positive integers arranged in a 'triangle'. These numbers are to be rearranged so that the five integers along each 'edge' of the triangle have the same sum, unlike the example shown.
When this is done, what is the greatest possible such sum?

A 38 B 42 C 48 D 52 E 54

5
10 1
7 13 14
3 8 2 12
15 4 9 6 11

SOLUTION **D**

It seems obvious that the greatest possible sum occurs when the three largest of the fifteen integers are at the vertices of the triangle, and the three smallest are in the interior of the triangle. However, for a complete solution, we need to prove this, and that an arrangement of this kind is possible.

Suppose that the integers from 1 to 15 are arranged in a triangle so that the sum of the numbers on each edge is S. We assume that in this arrangement the numbers at the vertices of the triangle are p, q and r, and the numbers in the interior of the triangle are u, v and w.

When we add up the numbers on each edge of the triangle, and then add up the three edge totals, we obtain the total of the fifteen integers, less u, v and w, and with p, q and r counted twice. Therefore

$$3S = (1 + 2 + \cdots + 14 + 15) - (u + v + w) + (p + q + r),$$

and therefore

$$S = 40 - \tfrac{1}{3}(u + v + w) + \tfrac{1}{3}(p + q + r).$$

[Here we have used the fact that $1 + 2 + \cdots + 14 + 15 = 120$.]

It follows that for S to have its largest possible value, $u + v + w$ has to be as small as possible, and $p + q + r$ has to be as large as possible. Therefore, to obtain the largest possible value for S, we seek an arrangement with 1, 2 and 3 in the interior of the triangle, and 13, 14 and 15 at the vertices of the triangle. The figure below shows that an arrangement of this kind is possible.

13
7 4
8 1 9
10 2 3 11
14 5 6 12 15

In this case

$$S = 40 - \tfrac{1}{3}(1 + 2 + 3) + \tfrac{1}{3}(13 + 14 + 15) = 40 - 2 + 14 = 52.$$

It follows that the greatest possible sum is 52.

FOR INVESTIGATION

25.1 (a) Find a formula for the sum of the first n positive integers.

(b) Use this formula to verify that $1 + 2 + \cdots + 14 + 15 = 120$.

(c) Find the sum of the first 1000 positive integers.