United Kingdom
Mathematics Trust

Junior Mathematical Olympiad

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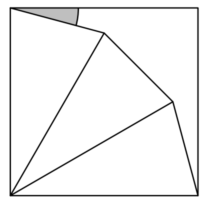
Solutions

A1. Evaluate $\left(1+\frac{1}{1^2}\right)\left(2+\frac{1}{2^2}\right)\left(3+\frac{1}{3^2}\right)$.

SOLUTION **14**

$\left(1+\frac{1}{1^2}\right)\left(2+\frac{1}{2^2}\right)\left(3+\frac{1}{3^2}\right)=\left(1+\frac{1}{1}\right)\left(2+\frac{1}{4}\right)\left(3+\frac{1}{9}\right)=2\times\frac{9}{4}\times\frac{28}{9}=14.$

A2. Three identical isosceles triangles fit exactly (without overlap) into a square, with two of the triangles having edges in common with the square, as shown in the diagram. What is the size of the shaded angle?



SOLUTION 15°

As the three isosceles triangles are identical, the three angles which meet in the bottom left-hand corner of the square are all equal. So each one is $90° \div 3 = 30°$. In each triangle, the other two angles are equal. Hence each is $(180-30)° \div 2 = 75°$.
Therefore the shaded angle is $(90-75)° = 15°$.

A3. What is the value of $\left(\frac{4}{5}\right)^3$ as a decimal?

SOLUTION **0.512**

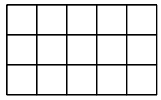
$\left(\frac{4}{5}\right)^3=\frac{4^3}{5^3}=\frac{64}{125}=\frac{64\times8}{125\times8}=\frac{512}{1000}=0.512.$

A4. In the first week after his birthday, Bill's uncle gave him some money for his piggy bank. Every week after that Bill put £2 into his piggy bank. At the end of the ninth week after his birthday Bill had trebled the amount he started with. How much did he have in total at the end of the ninth week?

SOLUTION **£24**

Let the amount Bill's uncle gave him in the first week after his birthday be £x. Every week after that Bill put £2 into his piggy bank and so by the end of the ninth week Bill had added £16 in total to his piggy bank. Therefore, the total amount in his piggy bank at the end of the ninth week was £$(x+16)$. Also, by the end of the ninth week, Bill had trebled the amount he started with and so $(x+16)=3x$, which gives $x=8$.
Therefore, Bill had £24 in total at the end of the ninth week.

A5. Aston wants to colour exactly three of the cells in the grid shown so that the coloured grid has rotational symmetry of order two. Each of the cells in the grid is a square. In how many ways can Aston do this?



Solution **7**

As the number of cells to be coloured is odd, one of those must map onto itself when the rectangle is rotated through 180° about its centre. So the central cell in the rectangle must be coloured. The diagram shows that there are seven pairs of cells which may be chosen as the second and third cells to be coloured so that the coloured grid has rotational symmetry of order 2. So Aston can complete the grid in seven ways.

a	*b*	*c*	*g*	*f*
d	*e*		*e*	*d*
f	*g*	*c*	*b*	*a*

A6. To travel the 140 kilometres between Glasgow and Dundee, John travels half an hour by bus and two hours by train. The train travels 20 km/h faster than the bus. The bus and the train both travel at constant speeds. What is the speed of the bus?

Solution **40 km/h**

Let the speed of the bus be v km/h. Then the speed of the train is $(v + 20)$ km/h.
Therefore $\frac{1}{2} \times v + 2 \times (v + 20) = 140$. So $\dfrac{5v}{2} + 40 = 140$. Hence $v = \dfrac{2 \times 100}{5} = 40$.

A7. The product of five different integers is 12. What is the largest of the integers?

Solution **3**

The smallest product of five different positive integers is $1 \times 2 \times 3 \times 4 \times 5 = 120$.
Therefore, at least one of the integers in this case is negative. Also, as the product of the integers is positive, the number of negative integers amongst the five different integers is even.
The smallest positive product of four different integers is $-2 \times -1 \times 1 \times 2 = 4$.
So the only five different integers whose product is 12 are $-2, -1, 1, 2$ and 3.

A8. In my desk, the number of pencils and pens was in the ratio 4 : 5. I took out a pen and replaced it with a pencil and now the ratio is 7 : 8. What is the total number of pencils and pens in my desk?

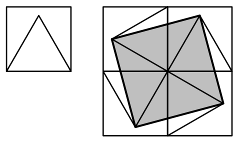
SOLUTION **45**

As the original ratio of the number of pencils to the number of pens was 4 : 5, let the original number of pencils be $4x$. Then the original number of pens was $5x$. When a pen is replaced with a pencil, the ratio of pencils to pens becomes 7 : 8.
Therefore

$$\begin{aligned} \frac{4x+1}{5x-1} &= \frac{7}{8} \\ \Rightarrow 7(5x-1) &= 8(4x+1) \\ \Rightarrow 35x-7 &= 32x+8 \\ \Rightarrow 3x &= 15 \\ \Rightarrow x &= 5. \end{aligned}$$

Therefore, the total number of pencils and pens in my desk is $4 \times 5 + 5 \times 5 = 45$.

A9. A unit square has an equilateral triangle drawn inside it, with a common edge. Four of these squares are placed together to make a larger square. Four vertices of the triangles are joined up to form a square, which is shaded and shown in the diagram. What is the area of the shaded square?



SOLUTION **2**

First note that the side-length of the equilateral triangle is equal to the side-length of the unit square, that is 1. So the diagonals of the shaded square have length 2.
Let the side-length of the shaded square be l.
Then, by Pythagoras' Theorem, $l^2 + l^2 = 2^2$. So $l^2 = 2$.
Hence the area of the shaded square is 2 square units.

A10. Amy, Bruce, Chris, Donna and Eve had a race. When asked in which order they finished, they all answered with a true and a false statement as follows:

Amy: Bruce came second and I finished in third place.

Bruce: I finished second and Eve was fourth.

Chris: I won and Donna came second.

Donna: I was third and Chris came last.

Eve: I came fourth and Amy won.

In which order did the participants finish?

Solution **Bruce, Donna, Amy, Eve, Chris**

For convenience, let A1 denote Amy's first statement, B2 Bruces's second statement, and so on.

Suppose A1 is true so that **Bruce was second**. Then B1 is true and B2 is false. Then E1 is false and hence E2 is true, so that **Amy won**. However C1 and C2 are now both false, which is a contradiction.

Hence A1 is false so that A2 is true, i.e. **Amy was third**. Also, since B1 is now false, B2 is true so that **Eve was fourth**. Note that D1 is false, as Amy was third. So D2 is true and **Chris came last**. C1 is now false and so C2 is true, i.e. **Donna came second**. Finally, we deduce that **Bruce won**, giving the order **Bruce, Donna, Amy, Eve, Chris**.

(*It is left to the reader to check that this order is consistent with each of the participants having given one true statement and one false statement.*)

Section B

B1. The solution to each clue of this crossnumber is a two-digit number, that does not begin with a zero.

Across	Down
1. A prime	1. A square
3. A square	2. A square

1	2
3	

Find all the different ways in which the crossnumber can be completed correctly.

Solution

First note that the only units digits of two-digit squares are 1, 4, 5, 6 and 9. As 3 across is a square composed of two such digits, its only possible values are 16, 49 and 64. If 3 across is 16, then 1 down is 81 and 2 down is either 16 or 36. In turn, 1 across is 81 or 83. Of these, only 83 is prime so there is exactly one way of completing the crossnumber when 3 across is 16.

[1] 8	[2] 3
[3] 1	6

Alternatively, if 3 across is 49 then 1 down is 64 and 2 down is 49. However, this makes 1 across 64, which is not prime.

Finally, if 3 across is 64 then 1 down is 16 or 36 and 2 down is 64. This means 1 across is 16 or 36, neither of which is prime.

Therefore there is exactly one way of completing the crossnumber, as shown.

B2. The Smith family went to a restaurant and bought two Pizzas, three Chillies and four Pastas. They paid £53 in total.

The Patel family went to the same restaurant and bought five of the same Pizzas, six of the same Chillies and seven of the same Pastas. They paid £107 in total.

How much more does a Pizza cost than a Pasta?

Solution

Let the prices in pounds of one Pizza, one Chilli and one Pasta be x, y, z respectively. Then, $2x + 3y + 4z = 53 \ldots [1]$. Also, $5x + 6y + 7z = 107 \ldots [2]$.

$[1] \times 2 : 4x + 6y + 8z = 106 \ldots [3]$.

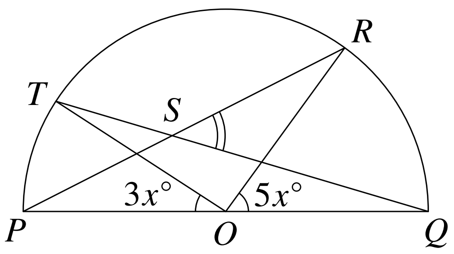
$[2] - [3] : x - z = 1$.

So a Pizza costs £1 more than a Pasta.

B3. Two overlapping triangles POR and QOT are such that points P, Q, R and T lie on the arc of a semicircle of centre O and diameter PQ, as shown in the diagram.

Lines QT and PR intersect at the point S. Angle TOP is $3x°$ and angle ROQ is $5x°$.

Show that angle RSQ is $4x°$.



SOLUTION

Triangle OPR is isosceles with $OP = OR$, as both sides are radii of the circle.
Therefore $\angle RPO = \angle PRO$. Also, applying the exterior angle theorem to triangle OPR, we note that $\angle ROQ = \angle RPO + \angle PRO$.
Hence $\angle RPO = (\frac{5x}{2})°$. Similarly, $\angle TQO = (\frac{3x}{2})°$.
Finally, applying the exterior angle theorem to triangle SPQ, we see that
$\angle RSQ = \angle SPQ + \angle SQP = (\frac{5x}{2})° + (\frac{3x}{2})° = 4x°$.

B4. The letters A, B and C stand for different, non-zero digits.

Find all the possible solutions to the word-sum shown.

$$\begin{array}{r} A\ B\ C \\ B\ C\ A \\ +\ C\ A\ B \\ \hline A\ B\ B\ C \end{array}$$

SOLUTION

Looking at the units column, we can deduce that, as A and B are non-zero and $A + B < 20$, $C + A + B = 10 + C$. So $A + B = 10$ and there is a carry of 1 to the tens column. In this column, as A and C are non-zero and $A + C < 19$, $1 + B + C + A = 10 + B$. Hence $C + A = 9$ and there is a carry of 1 to the hundreds column. The calculation in the hundreds column is the same as that in the tens column and there is a carry of 1 to the thousands column.
Therefore $A = 1$, $B = 10 - 1 = 9$ and $C = 9 - 1 = 8$.
So there is exactly one possible solution to the word-sum: $198 + 981 + 819 = 1998$.

B5. In Sally's sequence, every term after the second is equal to the sum of the previous two terms. Also, every term is a positive integer. Her eighth term is 400.

Find the minimum value of the third term in Sally's sequence.

Solution

Let the first two terms of Sally's sequence be m and n respectively.
Then the next six terms of the sequence are $m + n$, $m + 2n$, $2m + 3n$, $3m + 5n$, $5m + 8n$ and $8m + 13n$. So $8m + 13n = 400$. Note that 400 is a multiple of 8.
Therefore, as 8 and 13 are coprime, n is a multiple of 8. Also, $n < \frac{400}{13} < 31$.
The table shows possible values of n and the corresponding values of m and the third term in the sequence, $m + n$.

m	n	$m + n$
37	8	45
24	16	40
11	24	35

As can be seen, the minimum value of the third term in Sally's sequence is 35.

B6. The integers 1 to 4 are positioned in a 6 by 6 square grid as shown and cannot be moved.

The integers 5 to 36 are now placed in the 32 empty squares. Prove that no matter how this is done, the integers in some pair of adjacent squares (i.e. squares sharing an edge) must differ by at least 16.

	1			2	
	3			4	

Solution

Note that each of the squares containing one of the integers from 1 to 4 inclusive has four adjacent squares, making a total of sixteen adjacent squares. The diagram shows one particular way of placing the integers from 5 to 19 inclusive in fifteen of these squares so that the integers in no pair of adjacent squares differ by 16 or more. However, no matter how this is done, the sixteenth adjacent square (marked with an asterisk in the diagram for the particular case) must now contain an integer greater than or equal to 20. Hence the integers in at least one pair of adjacent squares must differ by at least 16.

	5			7	
9	1	13	14	2	10
	16			17	
	6			8	
11	3	*	19	4	12
	15			18	