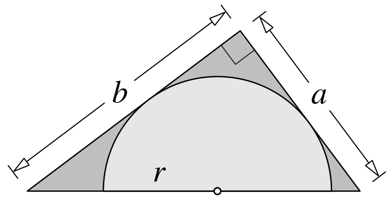
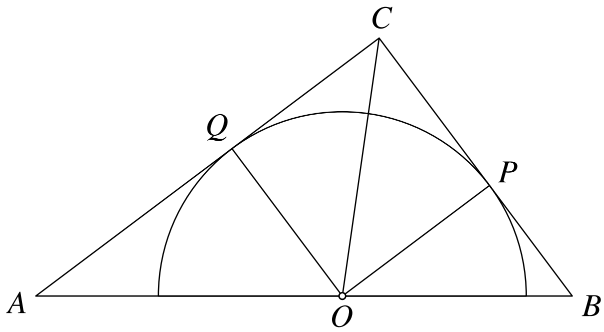
Maclaurin

1. The diagram shows a semicircle of radius r inside a right-angled triangle. The shorter edges of the triangle are tangents to the semicircle, and have lengths a and b. The diameter of the semicircle lies on the hypotenuse of the triangle.



Prove that
$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

Solution



Label the points as shown above, and join O to P, Q and C. Then OP is perpendicular to BC and OQ is perpendicular to CA, since each of OP and OQ is a radius and each of BC and CA is a tangent.

Method 1

The area of triangle AOC is $\frac{1}{2}rb$, that of triangle BOC is $\frac{1}{2}ra$ and that of triangle ABC is $\frac{1}{2}ab$.

Hence, adding, we have
$$\tfrac{1}{2}ab = \tfrac{1}{2}rb + \tfrac{1}{2}ra.$$

Dividing each term by $\frac{1}{2}abr$, we obtain
$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

Method 2

Since $OPCQ$ is a square, $AQ = b - r$. The triangles AOQ and ABC are similar (AA), so that

$$\frac{r}{b-r} = \frac{a}{b}.$$

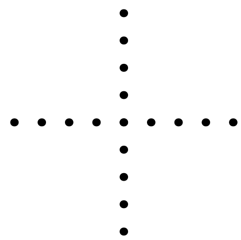
Multiplying each side by $b(b - r)$, we get

$$rb = ab - ar$$

and now, dividing each term by abr, we obtain

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

2. How many triangles (with non-zero area) are there with each of the three vertices at one of the dots in the diagram?



Solution

Method 1

There are 17 dots in the array, and we must choose 3 of them to obtain a triangle. This can be done in $\binom{17}{3} = 680$ ways. However, some of these triangles have zero area.

The triangle will have zero area if we choose all three dots on the same line. Hence the number of triangles of zero area is $2 \times \binom{9}{3} = 2 \times 84 = 168$.

So there are 512 triangles of non-zero area.

Method 2

We may choose two points on the line across the page in $\binom{9}{2} = 36$ ways, and one point on the line up the page (but not at the centre) in 8 ways. These choices give rise to $8 \times 36 = 288$ triangles of non-zero area.

Similarly we obtain another 288 triangles from two points on the line up the page and one on the line across the page.

But we have counted twice triangles with a vertex at the point where the lines meet, and there are $8 \times 8 = 64$ of these. So altogether we have $2 \times 288 - 64 = 512$ triangles.

3. How many solutions are there to the equation

$$m^4 + 8n^2 + 425 = n^4 + 42m^2,$$

where m and n are integers?

Solution

By rewriting the equation in the form

$$m^4 - 42m^2 + 425 = n^4 - 8n^2$$

and then completing the square on each side, we obtain

$$(m^2 - 21)^2 = (n^2 - 4)^2.$$

Taking the square root of each side, we get

$$m^2 - 21 = \pm(n^2 - 4).$$

Hence there are two cases to consider.

$\boldsymbol{m^2 - 21 = n^2 - 4}$

In this case, we have

$$m^2 - n^2 = 21 - 4,$$

so that

$$(m - n)(m + n) = 17.$$

Therefore, because 17 is prime, $m - n$ and $m + n$ are equal to 1 and 17, or -1 and -17, in some order.

Thus in this case there are four solutions for (m, n), namely $(\pm 9, \pm 8)$.

$\boldsymbol{m^2 - 21 = -(n^2 - 4)}$

In this case, we have

$$m^2 + n^2 = 21 + 4.$$

Hence

$$m^2 + n^2 = 5^2.$$

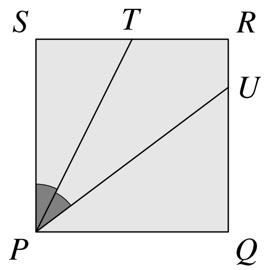
Now a square is non-negative, so that $-5 \leq m, n \leq 5$.

Thus in this case there are twelve solutions for (m, n), namely $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$ and $(\pm 4, \pm 3)$.

Therefore altogether there are sixteen solutions to the given equation.

4. The diagram shows a square $PQRS$ with sides of length 2. The point T is the midpoint of RS, and U lies on QR so that $\angle SPT = \angle TPU$.

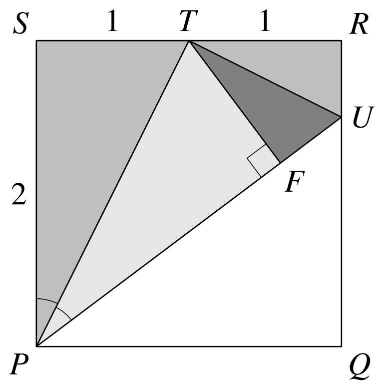
What is the length of UR?



Solution

Method 1

Let F be the point on PU so that $\angle TFP = 90°$ and join T to F and U, as shown. Then triangles PTS and PTF are congruent (AAS), so that $TF = 1$.



Hence triangles TUR and TUF are congruent (RHS), so that $\angle RTU = \angle UTF$.

Now the four angles at T are angles on the straight line RTS, so they add up to 180°. From the fact that $\angle TSP = 90°$ (because $PQRS$ is a square), we have

$$(90° - \angle SPT) + (90° - \angle TPF) + \angle RTU + \angle UTF = 180°.$$

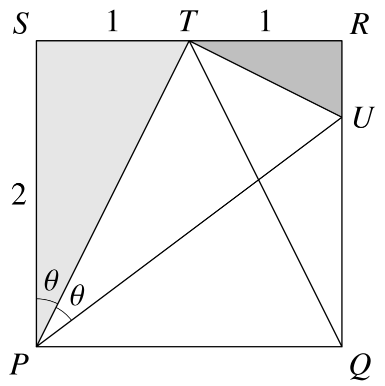
It follows that $\angle RTU = \angle SPT$.

Therefore triangles RTU and SPT are similar (AA), so that

$$\frac{UR}{RT} = \frac{TS}{SP} = \frac{1}{2}.$$

Thus $UR = \frac{1}{2}$.

Method 2



Let $\angle SPT$ and $\angle TPU$ both be equal to θ and join T to Q and U, as shown. Then triangles TSP and TRQ are congruent (SAS), so that $\angle TQR$ is also equal to θ.

From the converse of ‘angles in the same segment’ it follows that quadrilateral $UTPQ$ is cyclic. Hence, from ‘opposite angles of a cyclic quadrilateral’, we know that $\angle UTP$ is 90°, because $\angle PQR$ is 90°.

Now the four angles at T are angles on the straight line RTS, so they add up to 180°. It follows that $\angle RTU$ is equal to θ.

Therefore triangles RTU and SPT are similar (AA), so that

$$\frac{UR}{RT} = \frac{TS}{SP} = \frac{1}{2}.$$

Thus $UR = \frac{1}{2}$.

5. Solve the pair of simultaneous equations

$$(a+b)\left(a^2-b^2\right)=4 \quad \text{and}$$
$$(a-b)\left(a^2+b^2\right)=\frac{5}{2}.$$

SOLUTION

Since $a^2-b^2=(a-b)(a+b)$, we may write the first equation as $(a+b)^2(a-b)=4$.

Note that $a-b \neq 0$, since this would make the left-hand side of both equations zero, which is false. Hence we can divide the first equation by the second and cancel the term $a-b$ to produce

$$\frac{(a+b)^2}{a^2+b^2}=\frac{8}{5}.$$

Multiplying each side by $5\left(a^2+b^2\right)$, we get

$$5\left((a+b)^2\right)=8\left(a^2+b^2\right).$$

When we multiply this out and collect like terms, we obtain

$$\begin{aligned} 0 &= 3a^2-10ab+3b^2 \\ &= (3a-b)(a-3b), \end{aligned}$$

so either $a=3b$ or $b=3a$.

We substitute each of these in turn back into the first equation.

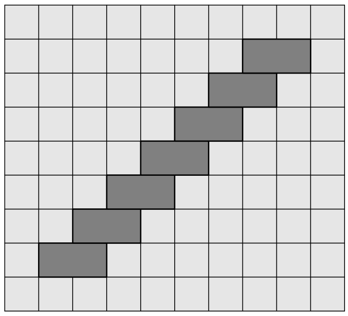
a* = 3*b
Then $4b \times 8b^2=4$, so that $b=\frac{1}{2}$ and $a=\frac{3}{2}$.

b* = 3*a
Then $4a \times \left(-8a^2\right)=4$, so that $a=-\frac{1}{2}$ and $b=-\frac{3}{2}$.

Hence we have two solutions $(a,b)=\left(\frac{3}{2},\frac{1}{2}\right)$ or $(a,b)=\left(-\frac{1}{2},-\frac{3}{2}\right)$. These solutions should be checked by substituting back into the second equation.

6. The diagram shows a 10×9 board with seven 2×1 tiles already in place.

What is the largest number of additional 2×1 tiles that can be placed on the board, so that each tile covers exactly two 1×1 cells of the board, and no tiles overlap?

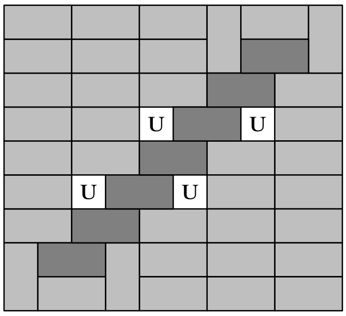


Solution

The first observation is that it is possible to add a further 36 tiles to the grid. The diagram shows one way of doing this: the additional tiles are lighter grey and the uncovered squares are indicated.

There are 36 additional tiles in the grid, and four squares are left uncovered.

We show that you cannot improve on this.



Colour the grid like a chessboard with alternating grey and white cells, as shown in Figure 1. Notice that any tile will cover one cell of each colour.

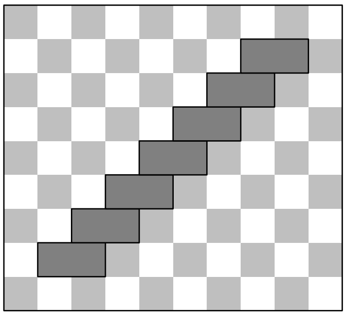


Figure 1

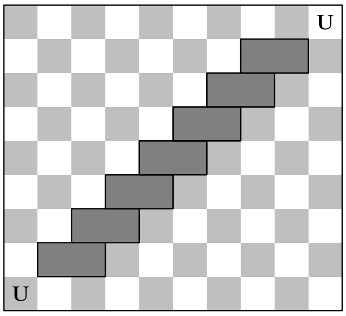


Figure 2

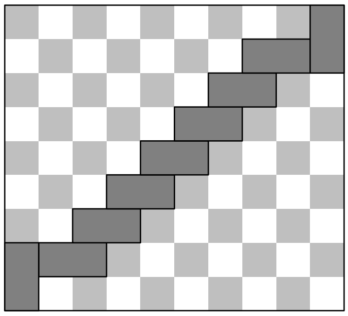


Figure 3

Suppose that two opposite corners are left uncovered, as shown in Figure 2. Then the remainder of the board consists of two separate 'staircases'.

The upper staircase has 17 grey and 20 white cells, so that at most seventeen 2×1 tiles may be placed there. Similarly for the lower staircase: at most seventeen 2×1 tiles may be placed there. In other words, with this arrangement, at most 34 additional tiles may be placed.

The only way to cover the corners whilst also reducing the 'excess' (the difference between the number of cells of one colour and the number of cells of the other) in both staircases is to place tiles in the corners as shown in Figure 3. This reduces the number of white cells in the upper staircase by one, and reduces the number of grey cells in the lower staircase by one. Once again at most seventeen tiles may be placed in each staircase, achieving at most 36 additional tiles in total.

Therefore the greatest number of additional 2×1 tiles that can be placed on the board is 36.