United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

6 – 8 October 2020

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Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: 2 hours.
3. Each question carries 10 marks.
4. Partial marks may be awarded for incomplete or partially correct answers. There are no negative marks for incorrect answers. You should therefore enter an answer even if you are not completely sure. However, one fully correct solution will gain more credit than several guesses.
5. Earlier questions tend to be easier. Some questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Follow the instructions on the answer sheet carefully. Hand the completed answer sheet to your teacher.
8. No answers may be filled in after the end of the 2 hours.
9. To accommodate candidates sitting on different days, please do not discuss the paper on the internet until 08:00 BST on 9th October 2020.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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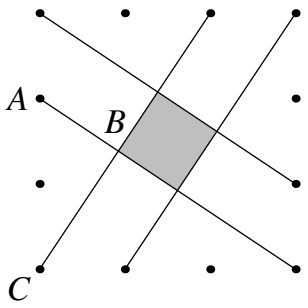
1. (a) Let p and q be different prime numbers, and let $a = p^2$, $b = pq$ and $c = p^3q$.
Which of ab, bc and ca is a square number? (1 mark)

Halle would like to place nine of the ten integers $1, 2, ..., 9, 10$ into the cells of a 3×3 grid in such a way that the three numbers within each row are written in increasing order from left to right and the product of the three numbers within each row is equal to a square number.

(b) Give an example showing that Halle's task is possible. (4 marks)

(c) How many different grids can Halle produce? (5 marks)

2. Twelve points, four of which are vertices, lie on the perimeter of a square. The distance between adjacent points is one unit. Some of the points have been connected by straight lines. B is the intersection of two of those lines, as shown in the diagram.



(a) Find the ratio $AB : BC$. Give your answer in its simplest form. (3 marks)

(b) Find the area of the shaded region. (7 marks)

3. (a) The expression $10xy - 2x + 5y - 1$ factorises as $(ax + b)(cy + d)$ where a, b, c, d are integers and $a > 0$. Find the values of a, b, c, d. (1 mark)

(b) Using the factorisation above and considering factor pairs of 99, find all integer pairs (x, y) such that $10xy - 2x + 5y = 100$.
Enter y values *only* on the answer sheet. (4 marks)

(c) Find all integer pairs (x, y) such that $10xy - 2x + 5y = 100000001$.
Enter y values *only* on the answer sheet. (5 marks)

4. Daniel and Alessia each have eight boxes labelled 1 to 8. To start, each of them has n balloons, all in their box 1.

(a) Alessia is playing a game with her balloons. She may make the following move:

- Choose a box k containing at least two balloons. Pop a single balloon in box k and then move another balloon from box k to box $k + 1$.

Find the value of n such that, after a finite number of moves, all that remains is a single balloon in box 8. (2 marks)

(b) Daniel is playing a *different* game with his balloons. He may use any combination of the following two moves:

- Pop two balloons in box k and move a third balloon from box k to box $k + 1$.
- Pop a single balloon in each of boxes k and $k + 1$ and then move another balloon from box $k + 1$ to box $k + 2$.

What is the *smallest* value of n such that after a finite number of moves, all that remains

(i) is a single balloon in box 4?

(ii) is a single balloon in box 8? (8 marks)

5. Freya creates a sequence with first term 1 and each subsequent term 5 more than the previous term. Hilary creates a different sequence with first term a and each subsequent term 3 less than the previous term. Both sequences are continued forever.

(a) There is at least one number that appears in both sequences. Let c be the smallest of those numbers. Find the possible values of c. (3 marks)

(b) Given that there are exactly 100 numbers which appear in both sequences, find the possible values of a.

Enter the following on the answer sheet:

(i) How many possible values of a are there?

(ii) The smallest possible value of a.

(iii) The largest possible value of a.

(7 marks)

