United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

Wednesday 28 September 2022

Organised by the United Kingdom Mathematics Trust

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Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks.
4. Questions 2 and 3 require answers only. The spaces for answers are clearly indicated on the answer sheets.
5. Questions 1, 4 and 5 require full written explanations. If your solution involves calculations, equations, tables, etc., explain where these come from and how you are using them. Explain how the steps of your solution link together, and give full proofs of assertions that you make. Answers alone will gain few marks (if any).
6. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.
   However, one complete solution will gain more credit than several unfinished attempts.
7. Earlier questions tend to be easier. Questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
8. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
9. You may use rough paper to note down your ideas, but you should write up your solution on the answer sheet provided for each question.
10. Start each question on an official master answer sheet that has a QR code on.
    You may use additional sheets (blank or lined paper only). On each additional sheet please write the number of the question in the top left-hand corner, followed by the QR code digits following the ':' symbol. Please do not write your name or initials on additional sheets.
11. Write on one side of the paper only.
12. Arrange your answer sheets in question order before they are collected. Please remove blank answer sheets i.e. those you do not wish to submit a solution for.
13. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Thursday 29 September.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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www.ukmt.org.uk

1. This question requires full written explanations.

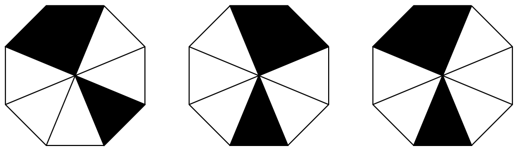
The points A, B and C lie, in that order, on a straight line. Line CD is perpendicular to AC, and $CD = AB$. The point E lies on the line AD, between A and D, so that $EB = EC = AB$.

(a) Draw a diagram to show this information. Your diagram need not be accurate or to scale, but you should clearly indicate which lengths are equal. (2 marks)

(b) Calculate the size of the angle BAE. (8 marks)

2. This question requires answers only.

In this question, two figures are considered to be different-looking if one cannot be rotated to produce the other. For example, in the diagram below, the first two figures are *not* different-looking, but the third one is different-looking from the first two.



(a) I have lots of congruent square tiles. Half of them are painted white and the other half are painted black. I fit four of these tiles together to make a larger square.

(i) Draw the two different-looking squares I can make using two white and two black tiles. (You can either colour your squares, or use the letters W and B to indicate colours.)

(ii) How many different-looking squares can I make in total?

(3 marks)

(b) I have lots of congruent tiles, each in the shape of an equilateral triangle. Half of them are painted white and the other half are painted black. I fit six of these tiles together to make a regular hexagon.

(i) How many different-looking hexagons can I make using three white and three black tiles?

(ii) How many different-looking hexagons can I make in total?

(7 marks)

3. This question requires answers only.

Define $f(x)$ to be the integer part of $\sqrt[3]{x}$; for example $\sqrt[3]{3.375} = 1.5$ so $f(3.375) = 1$, $\sqrt[3]{9} \approx 2.08$ so $f(9) = 2$, and $\sqrt[3]{27} = 3$ so $f(27) = 3$.

(a) Write down the first six positive cube numbers. Hence write down the value of $f(122)$.

(1 mark)

(b) If x is a positive integer with $f(x) = 3$, find the possible values of $f(2x)$. (3 marks)

(c) Find all positive integer values of x such that $f(x) + f(2x) + f(3x) = 10$. (6 marks)

4. This question requires full written explanations.

Freya and Hilary play a game. Freya first chooses a positive integer a, with $1 \leq a \leq 2022$. Then Hilary chooses a positive integer b in response, with $1 \leq b \leq 2022$, where b may equal a.

Next they consider the sequence with nth term given by $an + b$ (for $n = 1, 2, 3 \ldots$). If at least one term in the sequence is a multiple of ten then Freya wins the game and if not Hilary wins the game.

(a) Explain why, if Freya chooses $a = 2017$, Hilary cannot win the game. (1 mark)

(b) If Freya chooses $a = 2015$, for how many values of b will Hilary win the game?

(2 marks)

(c) For how many values of a is it guaranteed that Freya will win the game, no matter Hilary's choice of b? (7 marks)

You should make it clear which values of a are included in your count, why Freya always wins for those values of a, and how Hilary can win for all other values of a.

5. This question requires full written explanations.

(a) Given that m is a positive integer,

(i) What are the possibilities for the last digit of 2^m?

(ii) What are the possible remainders when 2^m is divided by 3?

(2 marks)

(b) Find all positive integer values of n, a and b, with $a \leq b$, such that $n! = 2^a + 2^b$. Justify carefully why there are no other possibilities.

(8 marks)

Note: For a positive integer n we define $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

