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Mathematical Olympiad for Girls 2014

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which shows no sign of being a genuine solution is marked on a '0 plus' basis; up to 3 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

This was the second year of the new format of the Mathematical Olympiad for Girls, in which some questions are split into two parts. The purpose of the first part is to introduce results or ideas needed to answer the second part.

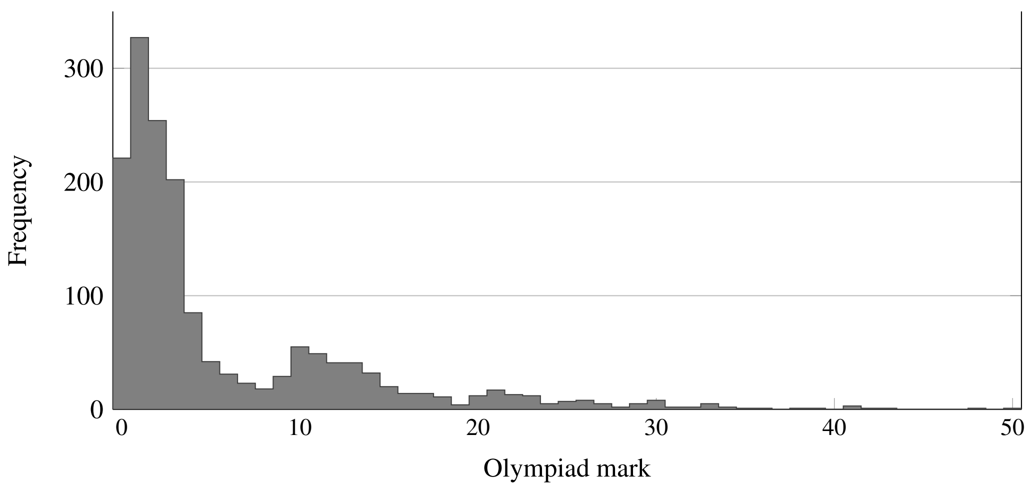
We were pleased that most candidates had a reasonable attempt at all the questions. Many completed part (a) of more than one question. It was also good to see a lot of candidates making a decent attempt to explain and justify their solutions: they grasped that a single numerical answer would not suffice.

We saw many elegant and creative solutions. Many candidates demonstrated real mathematical potential, coming up with their own strategies to solve problems in clearly unfamiliar areas. There were candidates who achieved a high mark for the whole paper and, even more pleasingly, many who produced excellent solutions to individual questions.

One of the most common mistakes that candidates made was trying to argue from special cases, rather than realising that some generality was needed. This was most apparent in Question 3, where a large number of candidates considered only one particular order of deleting the symbols, and in Question 4, where some considered a rectangle or a parallelogram rather than a general trapezium.

The 2014 Mathematical Olympiad for Girls attracted 1630 entries. The scripts were marked on 4th and 5th October in Cambridge by a team of Ben Barrett, Andrew Carlotti, Lax Chan, Andrea Chlebikova, Philip Coggins, Tim Cross, Susan Cubbon, Paul Fannon, James Gazet, Adam P. Goucher, Jo Harbour, Maria Holdcroft, Ina Hughes, Freddie Illingworth, Magdalena Jasicova, Vesna Kadelburg, Lizzie Kimber, David Mestel, Joseph Myers, Vicky Neale, Peter Neumann, Sylvia Neumann, Craig Newbold, Martin Orr, Preeyan Parmar, David Phillips, Linden Ralph, Jenni Sambrook, Eloise Thuey, Jerome Watson and Brian Wilson.

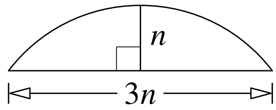
Mark distribution



Question 1

A chord of a circle has length $3n$, where n is a positive integer. The segment cut off by the chord has height n, as shown.

What is the smallest value of n for which the radius of the circle is also a positive integer?

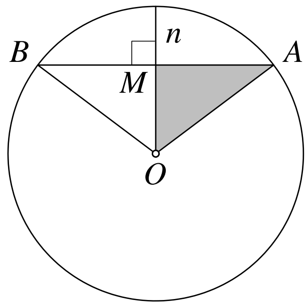


The question mentions the radius of the circle, so we should give it a name (such as r) in order to talk about it, and we should find it on the diagram. In fact it does not conveniently appear on the diagram given in the question, but if we draw the whole circle and its centre then we can find various interesting radii with length r.

It is important to remember that we are told that n and r are positive integers. Without this, it will not be possible to complete the problem.

Solution 10 marks

Let the centre and radius of the circle be O and r, and let A, B and M be the points marked in the following diagram.



Then $OA = OB = r$ and $OM = r - n$. Since the radius perpendicular to a chord bisects the chord, $AM = \dfrac{3n}{2}$. Now triangle OAM is right-angled, so we can use Pythagoras' Theorem to give

$$\begin{aligned} r^2 &= \left(\frac{3n}{2}\right)^2 + (r-n)^2 \\ &= \frac{9n^2}{4} + r^2 - 2rn + n^2 \end{aligned}$$

and hence

$$8rn = 13n^2.$$

Since n is positive and so non-zero, we can divide both sides by n to get

$$8r = 13n,$$

so that

$$r = \frac{13n}{8}.$$

For both r and n to be integers, n needs to be divisible by 8. But n is positive, so the smallest possible value of n is 8 (giving $r = 13$).

MARKERS' COMMENTS

We saw many excellent solutions to this problem; they all used the crucial idea of introducing the radius. They then went on to find some connection between r and n, perhaps by finding a useful triangle or using a circle theorem.

It is important to be clear about why 8 is the smallest possible value. We wrote r as

$$\frac{13n}{8}$$

and this makes it clear that n is a multiple of 8 (strictly speaking because the highest common factor of 8 and 13 is 1, although we did not require candidates to say this explicitly). Some candidates found (true) expressions such as $8r = 13n$ or $r = 1.625n$, but then it is not quite so obvious that n is a multiple of 8, so we needed candidates to explain how they knew that 8 is the *smallest* possible value, not just *a* possible value.

Question 2

(a) Some strings of three letters have the property that all three letters are the same; for example, LLL is such a string.

How many strings of three letters do not have all three letters the same?

(b) Call a number 'hexed' when it has a recurring decimal form in which both the following conditions are true.

(i) The shortest recurring block has length six.

(ii) The shortest recurring block starts immediately after the decimal point.

For example, $987.\dot{1}23\,45\dot{6}$ is a hexed number (the dots indicate that 123 456 is a recurring block).

How many hexed numbers are there between 0 and 1?

(a) We are asked to find the number of strings which do not have a certain property. One way to do this is to count how many strings do have the property, and then subtract this from the total number of strings.

It is also possible to count the number of strings with the given property directly. One way to do this is to think about possible starting combinations of letters: how many strings start with AA and how many with AB? Another way is to identify different possible types of strings — these are xyz, xxy, xyx and yxx, where x, y and z are different letters — and count each type separately. It helps to realise that the last three are going to give the same number. The last two approaches become considerably less practical when we are considering longer strings, such as in part (b). In general, excluding unwanted objects is often easier than counting the ones we want.

(b) We need to think about what a hexed number could look like. After the decimal point, it has six digits which then repeat. Since we are only looking for numbers between 0 and 1, the number before the decimal point is 0. So it looks like $0.abcdef\ abcdef\ \ldots$.

However, there are restrictions on what the recurring block $abcdef$ can be. The requirement that this should be the shortest recurring block means that it cannot have a shorter repeating block within it. For example, $0.12121212\ldots$ is not a hexed number because although the block '121212' repeats, the shorter block '12' also repeats. However, $0.121233\ 121233\ \ldots$ is a hexed number.

Now, counting all hexed numbers is the same as counting all possible blocks of length six which do not contain a shorter repeated sub-block. We can do this by identifying all the blocks that need to be excluded and taking this away from the total number of possible blocks of length six.

We need to exclude the following types of blocks:

(i) those where all six digits are the same;

(ii) those where a sub-block of two *different* digits repeats, such as *ababab* (blocks with all digits the same have already been excluded in (i));

(iii) those where a sub-block of three digits repeats, but not those where all three digits are the same (because these have already been excluded in (i)). Note that this is basically what we counted in part (a).

It is not possible to have sub-blocks of length 4 or 5, as those are not factors of 6.

The question does not specify whether 'between 0 and 1' includes the end-points. But, $0.\dot{0}$ and $0.\dot{9}$ are not hexed numbers, so they are going to be excluded anyway.

Solution to part (a) 2 marks

The total number of strings of three letters is 26^3, because there are 26 options for each letter. The number of strings where all three letters are the same is 26. So the number of strings which do not have all three letters the same is $26^3 - 26 = 17\,550$.

Solution to part (b) 8 marks

A hexed number between 0 and 1 is of the form $0.\dot{a}bcde\dot{f}$, where the block $abcdef$ contains no shorter repeating sub-block. Hence counting hexed numbers between 0 and 1 is the same as counting blocks of length six, made up of digits 0 to 9, which do not contain any repeated sub-blocks.

The total number of blocks of length six is 10^6, because there are 10 options for each digit. We need to exclude those blocks that have repeating sub-blocks of length one, two or three.

(i) If there is a repeating sub-block of length one, this means that all six digits are the same. There are 10 such blocks.

(ii) If there is a repeating sub-block of length two, the whole block looks like $ababab$, where a and b are different digits. There are 10 options for a and for each of them there are 9 options for b. So the number of blocks of the form $ababab$, where a and b are different, is $10 \times 9 = 90$.

(iii) The repeating sub-blocks of length three are precisely all strings of three digits except those where all three digits are the same. We essentially counted this in part (a) (although there we had 26 letters, rather than the 10 digits we have here), so the answer is $10^3 - 10 = 990$.

The total number of hexed numbers is therefore

$$10^6 - 10 - 90 - 990 = 998\,910.$$

Markers' comments

This was the most popular question, with almost half the candidates making substantial progress in part (a) and many attempting part (b) as well.

We saw several sensible approaches to calculating the total number of three-letter strings. Many candidates seemed familiar with the "multiplication rule" — counting the number of options for each letter and then multiplying those together — although some obtained 3^{26} instead of 26^3. Another common approach was to count the number of possibilities when the first letter is A, and then multiply the answer by 26. A common mistake here is to forget that the letters do not need to appear in alphabetical order; so having listed AAA, AAB, . . . , and then having moved onto the strings starting with AB, some candidates forgot to include ABA before ABB.

One advantage of counting all strings of three digits and subtracting the ones we don't want is that it easily extends to counting all possible strings of six digits, which we need in part (b).

The phrase "the shortest recurring block has length six" seemed to confuse some candidates, who interpreted it as meaning that a hexed number can have recurring blocks of any length greater than or equal to six. What it means is that there is a recurring block of length six, and no shorter recurring block.

We were happy to accept correct expressions instead of numerical values, so answers in the form $26^3 - 26$ or $25 \times 26 \times 27$ received full marks (even where candidates had attempted the calculation and made a slip in the arithmetic). We also accepted answers from candidates who used alphabets with different numbers of letters (not necessarily 26), or thought that there were nine digits rather than ten, as long as the answer was correct for their preferred choice.

Question 3

A large whiteboard has 2014 + signs and 2015 − signs written on it. You are allowed to delete two of the symbols and replace them according to the following two rules.

(i) If the two deleted symbols were the same, then replace them by +.
(ii) If the two deleted symbols were different, then replace them by −.

You repeat this until there is only one symbol left. Which symbol is it?

It may be useful to start investigating what happens by looking at a particular order of deleting the symbols. For example, we could delete 1007 pairs of − signs, leaving only one − sign and 3021 + signs (because (2014 + 1007 = 3021). Then keep deleting pairs of + signs until there is only one left. This leaves one + and one − sign, and deleting those leaves you with a − sign.

There are several other examples leading to the same answer. Another useful approach may be to investigate what happens if you start with fewer symbols, but we should think carefully about the significance of the specific numbers 2014 and 2015.

At this point we might think that we have solved the problem. However, there is a very common trap here! The description above shows that we *can* end up with one − sign. But it does not guarantee that this will happen *regardless of the order in which you delete the symbols*. To solve the problem, we must show either that we always end up with one − sign, or that that the final symbol depends on the order in which we deleted them.

We need an argument that does not rely on a specific example. The answer probably has something to do with the fact that there is an odd number of − signs, which seems to make it impossible to eliminate the final one. So one approach is to look at whether, after each step, the number of each type of symbol is even or odd (this is called the *parity* of the number).

Alternatively, the rules of the game should remind us of the rules of multiplication of positive and negative numbers. We present two solutions, one using each of these ideas.

Solution 10 marks

METHOD 1

At each step, one of the following three things happens:

(i) two + signs are replaced by one + sign;
(ii) two − signs are replaced by one + sign;
(iii) one + sign and one − sign are replaced by one − sign.

In the first and the third case the number of − signs remains the same, while in the second case it decreases by two. Since at the beginning the number of − signs was odd, it will remain odd after each step. Hence we can never eliminate all the − signs, so this is the last remaining symbol.

METHOD 2

If we replace every + sign with the number 1 and every − sign with the number −1, then the rules are equivalent to replacing a pair of numbers with their product. The final number remaining is the product of all the numbers we started with, which is $1^{2014} \times (-1)^{2015} = -1$. Hence the last remaining symbol is a − sign.

MARKERS' COMMENTS

This was the second most popular question and more than half the candidates scored marks on it.

A large number of candidates did not realise the need for generality and may be disappointed with the number of marks they received. However, we did see many elegantly presented solutions, and many more attempts to explain observed patterns. Some noticed the constant parity of the number of minus signs without realising that this essentially solves the problem. Many thought that the key issue is that the number of minus signs is one more than the number of plus signs; this is not the case.

Another important point to note here is the need for clear presentation and commentary. A table containing the numbers of plus and minus signs at various stages can be difficult to interpret. It is much better to describe the strategy, such as saying which of the three rules is being used at each stage.

Question 4

(a) In the quadrilateral $ABCD$, the sides AB and DC are parallel, and the diagonal BD bisects angle $\angle ABC$. Let X be the point of intersection of the diagonals AC and BD. Prove that $\dfrac{AX}{XC} = \dfrac{AB}{BC}$.

(b) In triangle PQR, the lengths of all three sides are positive integers. The point M lies on the side QR so that PM is the internal bisector of the angle $\angle QPR$. Also, $QM = 2$ and $MR = 3$.

What are the possible lengths of the sides of the triangle PQR?

(a) The first thing to do when facing a geometry problem is to draw and label a clear diagram. The purpose of a diagram is to convey information, so be sure to draw diagrams clearly, don't make them too small, and label them carefully. It was pleasing to see that most candidates did this as their first step.

It is important not to make additional assumptions that are not explicitly stated in the question. For example, we are told that AB and DC are parallel, but nothing about BC and AD; you should therefore draw a trapezium and not a rectangle or a parallelogram.

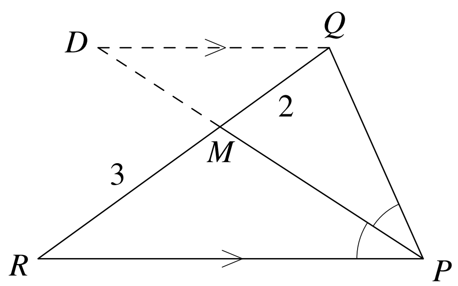
You should note the convention for labelling quadrilaterals: $ABCD$ means that the vertices should appear in that order when going around the shape.

The fact that you are being asked to prove something about ratios of sides should make you think about similar triangles. To find two similar triangles, we need to look at angles. We are told about the angle bisector, and we also have some parallel lines, so there are some good starting points in looking for equal angles. Once we have identified some equal angles, two useful questions to ask are:

(i) are there any similar triangles?
(ii) are there any isosceles triangles?

There are several alternative ways to prove this result, for example by using the sine rule or areas of triangles ABX and BXC. This does not use point D at all. In fact, the result in part (a) is really about triangle ABC and it is called the *Angle Bisector Theorem*. It tells us something about the ratio in which an angle bisector divides the opposite side.

(b) The set-up looks similar to part (a), in that there is an angle bisector and the point at which it intersects the opposite side of the triangle. So you should be looking to use the result from part (a). If you cannot immediately see how, you can add another point to the diagram so that it looks like the one from part (a). To do this, you need to extend the line PM to a point D such that DQ is parallel to RP.



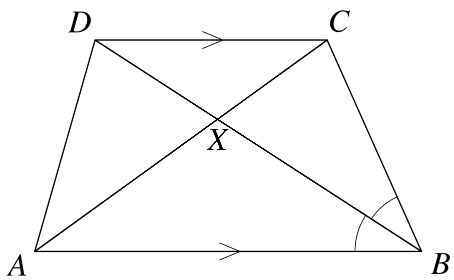
You should now be able to see that part (a) applies, so that

$$\frac{3}{2} = \frac{RP}{PQ}.$$

We now have the ratio of the lengths of two sides of the triangle. Remembering that they need to be integers, you can deduce that PQ has to be an odd number and RP a multiple of 3. But this still leaves infinitely many possibilities. Is there any property of a triangle that means that only some of these are actually possible?

In any triangle, the sum of the lengths any two sides is greater than the length of the third side; this is called the *triangle inequality*. Combining it with the ratio we found above leads to a finite number of possible combinations of sides.

Solution to part (a) 3 marks



Since AB is parallel to DC, $\angle ABD = \angle CDB$ (alternate angles). In triangles ABX and CDX, the angles at X are also equal (vertically opposite angles). Hence the two triangles are similar, and

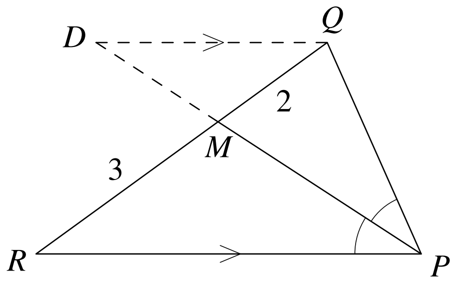
$$\frac{AX}{CX} = \frac{AB}{CD}.$$

We are also given that $\angle ABD = \angle DBC$, so that $\angle CDB = \angle DBC$. This implies that $BC = CD$ (sides opposite equal angles in triangle BCD). Combining this with the above, we get

$$\frac{AX}{XC} = \frac{AB}{BC},$$

as required.

Solution to part (b) 7 marks



Using the result from part (a) (as shown in the diagram), we get

$$\frac{RP}{PQ} = \frac{3}{2}.$$

But PQ, RP and 5 are sides of a triangle, so they satisfy the three triangle inequalities

$$\begin{aligned} PQ + RP &> 5, \\ PQ + 5 &> RP \\ \text{and} \quad RP + 5 &> PQ. \end{aligned}$$

Substituting $RP = \frac{3}{2}PQ$, we get

$$\begin{aligned} \tfrac{5}{2}PQ &> 5, \\ PQ + 5 &> \tfrac{3}{2}PQ \\ \text{and} \quad \tfrac{3}{2}PQ + 5 &> PQ. \end{aligned}$$

The third inequality is always true, but from the first two inequalities we obtain $PQ > 2$ and $PQ < 10$. Since PQ is an integer, the possible values are 3, 4, 5, 6, 7, 8 and 9. However, $RP = \frac{3}{2}PQ$ is also an integer, so PQ is even. Therefore there are only three possibilities:

$$\begin{aligned} PQ = 4,\ RP = 6 \text{ and } QR = 5; \\ PQ = 6,\ RP = 9 \text{ and } QR = 5; \\ PQ = 8,\ RP = 12 \text{ and } QR = 5. \end{aligned}$$

MARKERS' COMMENTS

Given that students are often intimidated by geometry, it was great to see a large number of attempts at this question. Many candidates did realise the need for some generality and tried to avoid drawing squares and rectangles. Unfortunately we saw other, more subtle incorrect assumptions, such as: just because the diagonal BD bisects the angle at B, it also bisects the angle at D; or that the other diagonal will bisect angle A; or that the diagonals bisect each other. It is a useful exercise to try drawing some accurate diagrams to explore whether any of those are necessarily true; dynamic geometry software can be a very useful tool here.

This question revealed some common misconceptions about geometrical notation. For example, a number of candidates wrote statements like

$$\frac{AX}{BX} = \frac{A}{B},$$

suggesting that they thought of the letters used to label points as variables, so that AX represents the product of A and X. We saw similar mistakes when dealing with angles, such as '$\angle ABD = \angle DBC$ so $\angle A = \angle C$' ("cancelling" B and D). Since most senior Olympiad geometry problems do not come with a diagram, it is important that candidates can interpret geometrical notation and terminology correctly.

Some candidates were clearly familiar with the angle bisector theorem and simply quoted it in part (a), without giving a proof. As the intention of part (a) is to introduce potentially new concepts which can help in part (b), this was not penalised, as long as they clearly stated which triangle the theorem was being applied to.

In part (b) a fair number of candidates seemed familiar with the triangle inequality, but some did not realise that two of the inequalities were needed to limit the number of solutions. One common pitfall was to use the triangle inequality to *check* that $PQ = 4$, $PQ = 6$ and $PQ = 8$ work, but $PQ = 10$ does not, and conclude that all solutions have been found. This was penalised heavily, since this method does not *prove* that no larger value of PQ can work.

Having found the three possible pairs of sides, we should really check that they all satisfy the conditions of the question, that is, in each of the resulting triangles the angle bisector does indeed divide the side QR in the ratio $2 : 3$. This is easily done by using the converse of the angle bisector theorem but, since this was not included in the proof of the theorem in part (a), it was not required for full marks on this occasion.

Question 5

The AM-GM inequality states that, for positive real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

and equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

(a) Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

for all positive real numbers a, b and c, and determine when equality holds.

(b) Find the minimum value of

$$\frac{a^2}{b} + \frac{b}{c^2} + \frac{c}{a}$$

where a, b and c are positive real numbers.

(a) The statement of the AM-GM inequality looks rather intimidating. In part (a) there seem to be three numbers added together, so let's see what the inequality says when $n = 3$. If we call the three numbers x, y and z, then

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz},$$

where x, y and z are positive real numbers. It seems useful to set

$$x = \frac{a}{b},\ y = \frac{b}{c},\ z = \frac{c}{a}$$

AM-GM also tells us that equality holds if, and only if, $x = y = z$. (It is quite straightforward to check that if $x = y = z$ then both sides are x and so we have equality. What is true and useful, but much less obvious, is that this is the only way we get equality.)

(b) First of all, we need to understand what the question is asking us to do. We want to find the minimum value of the given expression, and this involves doing two things:

(i) finding a value that the expression is always greater than or equal to;
(ii) proving that this value is actually achieved for some a, b and c.

To see why step (ii) is required, think about these examples; x^2 is always greater than -1, but its minimum value is in fact 0; and for a positive number x, $\frac{1}{x}$ is always greater than 0, but it does not have a minimum value at all.

In part (a), we used the AM-GM inequality to show that a similar expression was always greater than or equal to 3, so it seems sensible to start with the same approach, applying AM-GM to $\frac{a^2}{b}$, $\frac{b}{c^2}$ and $\frac{c}{a}$. This gives

$$\begin{aligned}\frac{\frac{a^2}{b} + \frac{b}{c^2} + \frac{c}{a}}{3} &\geq \sqrt[3]{\frac{a^2}{b} \times \frac{b}{c^2} \times \frac{c}{a}} \\ &= \sqrt[3]{\frac{a}{c}}.\end{aligned}$$

Unfortunately, this last expression is not a constant, so it does not give us the minimum value.

It is tempting to hope that we can minimise the left-hand side by finding when equality occurs; it turns out that this is when $c = a$ and $b = a^2$, and then the right-hand side is 1. Unfortunately it might turn out that when $\sqrt[3]{\frac{a}{c}} < 1$ the left-hand side could be smaller than 1 while still being larger than the right hand side.

The problem seems to be that, this time, the as and cs do not all cancel. Is there a way we can rewrite the expression to make this happen?

Since there is an a^2 in the numerator of the first fraction, we would like two fractions with denominator a. We can achieve this by splitting the last fraction into two:

$$\frac{c}{a} = \frac{c}{2a} + \frac{c}{2a}.$$

This also gives us two cs in the numerator, to cancel the c^2 in the denominator of the second fraction. So we have written the expression as a sum of four fractions, in a way that allows us to apply the AM-GM inequality, as we did in part (a) (but to four quantities this time, rather than three).

There is an alternative approach: to apply the AM-GM inequality to the first two terms, and then again to the result and the third term. We present this as an alternative solution below.

Solution to part (a) 2 marks

Using the AM-GM inequality with $n = 3$ and setting $x_1 = \frac{a}{b}$, $x_2 = \frac{b}{c}$ and $x_3 = \frac{c}{a}$, we get

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}}.$$

The three fractions under the cube root multiply to 1, so this is equivalent to

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

Equality holds if and only if

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{a}.$$

Multiplying both sides of the first equation by b^2c, we get $abc = b^3$, and multiplying both sides of the second equation by ac^2, we get $abc = c^3$. Therefore $b^3 = c^3$, so that $b = c$, and thus $a = b$, from the first equation. Hence equality holds if and only if $a = b = c$.

Solution to part (b) 8 marks

Method 1

Rewrite the expression as

$$\frac{a^2}{b}+\frac{b}{c^2}+\frac{c}{2a}+\frac{c}{2a}.$$

Then, using the AM-GM inequality with $n=4$, we get

$$\begin{aligned}\frac{a^2}{b}+\frac{b}{c^2}+\frac{c}{2a}+\frac{c}{2a} &\geq 4\sqrt[4]{\frac{a^2}{b}\times\frac{b}{c^2}\times\frac{c}{2a}\times\frac{c}{2a}}\\ &= 4\sqrt[4]{\frac{1}{4}}\\ &= \frac{4}{\sqrt{2}}\\ &= 2\sqrt{2}.\end{aligned}$$

To show that $2\sqrt{2}$ is in fact the minimum value, we need to find at least one set of values of a, b and c for which the expression is equal to $2\sqrt{2}$. In the AM-GM inequality, equality holds if and only if all four numbers are equal, so we need

$$\frac{a^2}{b}=\frac{b}{c^2}=\frac{c}{2a}.$$

These are two equations in three variables, but we do not need to solve them — we are just trying to find values of a, b and c that work. So let us put $a=1$; the equations then become

$$\frac{1}{b}=\frac{b}{c^2}=\frac{c}{2}.$$

Multiplying both sides of the first equation by bc^2, we get $c^2=b^2$, so that $c=b$, since both b and c are positive. Now from the second equation we obtain $b=c=\sqrt{2}$.

We can check that these values do indeed work:

$$\begin{aligned}\frac{a^2}{b}+\frac{b}{c^2}+\frac{c}{a} &= \frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{1}\\ &= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\sqrt{2}\\ &= 2\sqrt{2}.\end{aligned}$$

Hence the minimum value of the required expression is $2\sqrt{2}$.

Method 2

Applying the AM-GM inequality (with $n=2$) to $\frac{a^2}{b}$ and $\frac{b}{c^2}$ gives

$$\begin{aligned}\frac{a^2}{b}+\frac{b}{c^2} &\geq 2\sqrt{\frac{a^2}{b}\times\frac{b}{c^2}}\\ &= \frac{2a}{c}.\end{aligned}$$

Applying AM-GM again (also with $n = 2$), this time to $2\frac{a}{c}$ and $\frac{c}{a}$, we get:

$$\begin{aligned}\frac{a^2}{b} + \frac{b}{c^2} + \frac{c}{a} &\geq \frac{2a}{c} + \frac{c}{a} \\ &\geq 2\sqrt{\frac{2a}{c} \times \frac{c}{a}} \\ &= 2\sqrt{2}.\end{aligned}$$

The minimum value is reached if and only if

$$\begin{aligned}\frac{a^2}{b} &= \frac{b}{c^2} \\ \text{and} \quad \frac{2a}{c} &= \frac{c}{a}.\end{aligned}$$

Setting $a = 1$, we obtain the same values of b and c from these equations that we found in the first method.

Remark

There are, of course, other values of a, b and c that also give the minimum value of the expression. We can solve the above equations to express b and c in terms of a: $c = \sqrt{2}a$ and $b = \sqrt{2}a^2$.

Markers' comments

It is not surprising that nearly half the candidates did not attempt this question at all, but we saw a good number of solutions to part (a). Only a handful of candidates made progress in part (b).

Several candidates did not realise that n stands for the number of variables involved, rather than the denominator of the fraction in general; they tried writing the left-hand side as $\dfrac{ab+bc+ca}{abc}$ and setting $n = abc$, which is incorrect.

Many candidates tried various values of a, b and c to see what happens. While this is useful in trying to understand the problem, it often led to statements like 'the minimum value is when a, b and c are the smallest possible, so $a = b = c = 1$'. But the question states that a, b, c are positive *real* numbers, so it is impossible to state the smallest possible values of a, b and c. A more subtle, but important, point is that the expression

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

is not necessarily made smaller by decreasing a, b or c. For example, when $a = b = c = 2$ the value of the expression is 3, but if we decrease a to 1 the value increases to $3\frac{1}{2}$.

In part (a), many candidates stated (correctly) that equality holds if $a = b = c$, but either did not know or did not mention that this is the only case when equality occurs. This can be deduced from AM-GM.

Interested students may wish to find out about other similar inequalities. AM-GM refers to the arithmetic mean and the geometric mean of a set of numbers. The arithmetic mean is what we often refer to as 'the mean': the sum of n numbers divided by n. The geometric mean is the nth root of their product. One interesting special case of AM-GM says that if x is a positive real number, then

$$x + \frac{1}{x} \geq 2.$$

Other related inequalities involve the quadratic mean (used in statistics) and the harmonic mean.