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Mathematical Olympiad for Girls 2022

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Introduction

For many students, and possibly some teachers, this is the first experience of attempting a Maths Olympiad paper. It may therefore be useful to understand how these papers are marked, as students may be disappointed to receive a small number of marks for a problem they thought they had almost solved.

Most questions in Olympiad papers are marked using what the call the '0+/10−' principle. This means that the markers first read the whole write-up and decide whether the student has a viable strategy to solve the problem. It may be that there are some mistakes or small gaps in their reasoning, but if those could be relatively easily filled in then this response is marked in the '10−' regime, with usually up to three marks being taken away for gaps and mistakes. Common examples of small gaps are algebraic or arithmetical errors (provided they don't change the nature of the argument), missing one of several cases in a counting question, or lack of geometrical reasons when calculating angles.

If, on the other hand, the student has only started to explore the problem and has only made some useful observations, but does not have a strategy to generalise or prove them, then the script is marked in the '0+' regime. Up to three marks may be available for spotting a pattern or trying an idea which, if progressed further, could lead to a solution. The example of the former in the present paper would be, in Question 5c, noticing that in order to make $2^a + 2^b$ a multiple of 3, one of a and b must be even and the other one odd. An example of the latter would be, in Question 1, labelling the required angle and expressing some other angles in terms of it. Notice that all those examples involve a substantial engagement with the problem, rather than just trying one or two examples. Teachers should therefore reiterate to students that scoring even one or two marks on any of these questions is a real achievement.

It is unfortunately often the case that students think that they have solved a problem but only receive two or three marks. The most common reason for this is that their solution relies on a series of unjustified claims. The prime example in this paper was Question 1, where many candidates (correctly but without justification) claimed that the line CE bisects the angle BCD. If those claims were purely made on the basis of observation or as a guess, then this does not constitute a proof and can only receive marks in the '0+' regime. Students in this situation are strongly advised to read these comments and the official solution, to understand how they can add sufficient detail to their proofs.

The Girls' Olympiad paper is slightly different from other Maths Olympiads in that questions are broken down into several parts. Most of the time, the final part is the "main question" and the first part (or parts) are intended to suggest some useful results or good approaches to the problem. The reason for structuring the paper in this way is that the setters know that many of the candidates are not experienced in olympiad mathematics, and the hope is that by giving these pointers, we enable them to engage with a question even if they are not familiar with some standard olympiad technique or "trick". A useful hint is to read the whole question first and try to understand how the early parts may be helpful in solving the main problem.

General comments

With two of the questions this year being answer-only, candidates had more time to focus on the remaining three questions, and this was evident in the improved quality of written communication in Questions 1, 4 and 5. This year also saw an over 40% increase in the number of entries.

Overall, the markers were impressed by the level of engagement with the paper. It was particularly pleasing to see so many candidates using hints from early parts of questions to solve some challenging problems.

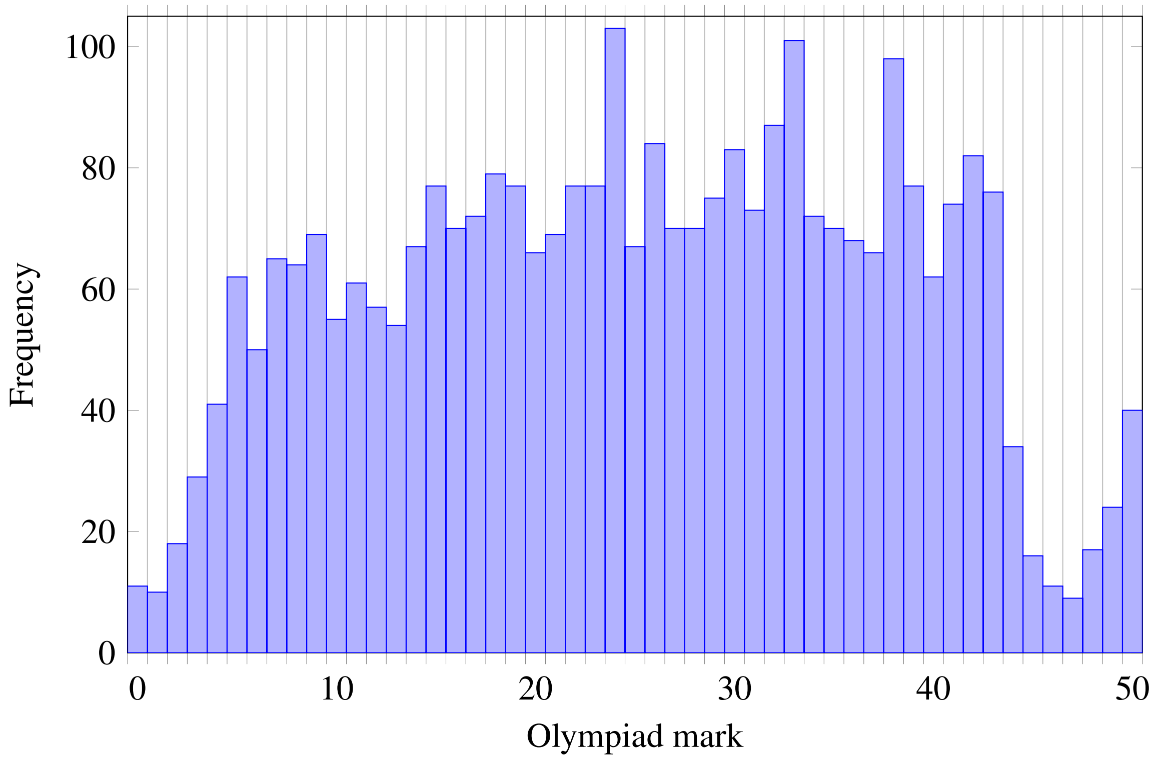
The highest scoring question was Question 1, despite being geometry, which is traditionally the least favourite olympiad topic. There were many well presented solutions with each step clearly justified. Students should be encouraged by this, and attempt geometry questions in future olympiads.

There were of course instances of solutions relying on unjustified assumptions, the most common being that the line CE bisects the right angle in Question 1 (which turns out to be true, but needs proving) and, in Question 5, that the fact that factorials grow faster than powers of 2 somehow means that the given equation can have no solutions. The latter is not a sufficient argument in itself – it should be clear that, for any given factorial, there is a power of 2 greater than it. Students should be encouraged to think critically about statements that they make, and test them using examples, to at least assess whether the statement could plausibly be true. For example, they could be asked to find two powers of 2 whose sum is smaller than 10!, and two whose sum is larger; or to find a power of 2 which is definitely greater than 100!.

Mark distribution

The MOG 2022 paper was marked by a team of Aleksander Goodier, Amit Goyal, Amit Shah, Andrea Chlebikova, Andrew Ng, Anne Andrews, Ben Handley, Cathy Beckett, Chris Garton, David Vaccaro, Dominic Rowland, Dominic Yeo, Ersel Awan, Eve Pound, Geoff Smith, James Cranch, Jeremy King, John Cullen, Joseph Myers, Kit Kilgour, Laura Daniels, Martin Orr, Melissa Quail, Michael Illing, Naomi Bowler, Oliver Murray, Phill Beckett, Przemysław Mazur, Richard Freeland, Robin Bhattacharyya, Sam Bealing, Stephen Tate, Thomas Lowe, Thomas Read, Vesna Kadelburg, Wendy Dersley and Wendy Rathbone.

We received non-empty scripts from 3086 candidates.



Question 1

The points A, B and C lie, in that order, on a straight line. Line CD is perpendicular to AC, and $CD = AB$. The point E lies on the line AD, between A and D, so that $EB = EC = AB$.

(a) Draw a diagram to show this information. Your diagram need not be accurate or to scale, but you should clearly indicate which lengths are equal. (2 marks)

(b) Calculate the size of the angle BAE. (8 marks)

SOLUTION

(See the official solutions document)

MARKERS' COMMENTS

This question did not require any knowledge beyond basic facts about angles sums in triangles and straight lines. The difficulty lay in coming up with a strategy, when none of the angles in the diagram were given.

Over half the candidates solved the problem, but more than a third of them lost some marks for lack of geometrical justification (i.e. stating which triangle or straight line are being used). Then mean mark was 6.4 and the median was 8.

The most successful approach was to set $x = \angle BAE$ then express other angles in terms of the single variable x. Filling these angles into the diagram as they calculated them was a great help to the candidate as well as the markers.

Using multiple variables x, y, z, etc led to systems of equations which were often solved incorrectly if at all. Using angle names to form equations such as $\angle AEB + \angle BEC + \angle CED = 180$ was even less successful, due to confusion between $\angle AEB$ and $\angle ABE$ for example.

Many candidates made false assumptions such as:

- ABE, BEC and ECD are congruent triangles because they have two common sides (the angles and third side are different in each case).
- BCE is equilateral (the question says nothing about distance BC, and it cannot be equal to the others). BE is parallel to CD and perpendicular to AB (it cannot be for the isosceles angles to work).

Other candidates made true assumptions such as:

- $\angle BEC = 90°$
- EC bisects angle ACD

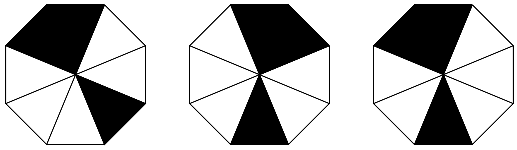
Although these statements happen to be correct, they are not given in the question and, in fact, proving them essentially requires solving the question. Thus solutions based on those assumptions were not rewarded.

Approaches using Pythagoras or Trigonometry end up requiring the use of more advanced trigonometry, such as double angle formulae, so were not useful to most candidates.

Question 2

This question requires answers only.

In this question, two figures are considered to be different-looking if one cannot be rotated to produce the other. For example, in the diagram below, the first two figures are *not* different-looking, but the third one is different-looking from the first two.



(a) I have lots of congruent square tiles. Half of them are painted white and the other half are painted black. I fit four of these tiles together to make a larger square.

(i) Draw the two different-looking squares I can make using two white and two black tiles. (You can either colour your squares, or use the letters W and B to indicate colours.)

(ii) How many different-looking squares can I make in total?

(3 marks)

(b) I have lots of congruent tiles, each in the shape of an equilateral triangle. Half of them are painted white and the other half are painted black. I fit six of these tiles together to make a regular hexagon.

(i) How many different-looking hexagons can I make using three white and three black tiles?

(ii) How many different-looking hexagons can I make in total?

(7 marks)

Solution

(See the official solutions document.)

Markers' comments

Almost all candidates attempted this question, and just under a third solved it fully. The mean mark was 5.9 and the median was 6.

Being an answer-only question, it is difficult to know what mistakes candidates made, but some of the most common incorrect answers give us a few clues.

- The answer to (a)(ii) was often given as 2, suggesting that candidates didn't realise there are shadings other than those with two squares of each colour.
- The answers to part (b) were often 3 and 13, or 2 and 12, rather than 4 and 14, suggesting that those candidates counted the colourings with three of each colour incorrectly, but then had a correct count for the rest of the cases. Such answers received partial marks.

Question 3

This question requires answers only.

Define $f(x)$ to be the integer part of $\sqrt[3]{x}$; for example $\sqrt[3]{3.375} = 1.5$ so $f(3.375) = 1$, $\sqrt[3]{9} \approx 2.08$ so $f(9) = 2$, and $\sqrt[3]{27} = 3$ so $f(27) = 3$.

(a) Write down the first six positive cube numbers. Hence write down the value of $f(122)$. (1 mark)

(b) If x is a positive integer with $f(x) = 3$, find the possible values of $f(2x)$. (3 marks)

(c) Find all positive integer values of x such that $f(x) + f(2x) + f(3x) = 10$. (6 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

This question had a very slightly lower mean mark than Question 2 (5.7 vs 5.9), but more fully correct answers, with just over a third of candidates scoring full marks. The median was 7 marks, a whole mark higher than Question 2.

Again, it is difficult to tell what thought process led to incorrect answers, but some common mistakes were:

- Dealing incorrectly with the end of the interval in part (c), most commonly including 32 and sometimes excluding 27. In some cases this may have been the mistake with notation, i.e. writing ≤ 32 when < 32 was intended.
- Including numbers up to 41. This includes all the values for which $f(3x) = 4$, but between 32 and 41 we have $f(2x) = 4$ as well, so the three function values sum to 11 rather than 10.
- In part (b), some candidates gave the answer as "between 54 and 126", which are the correct values of $2x$, rather than $f(2x)$.

All of those answers received partial marks. One the other hand, candidates who carelessly wrote down five or seven cube numbers, or included 0 as a positive cube number, did not receive a mark for part (a).

Question 4

This question requires full written explanations.

Freya and Hilary play a game. Freya first chooses a positive integer a, with $1 \leq a \leq 2022$. Then Hilary chooses a positive integer b in response, with $1 \leq b \leq 2022$, where b may equal a.

Next they consider the sequence with nth term given by $an + b$ (for $n = 1, 2, 3 \ldots$). If at least one term in the sequence is a multiple of ten then Freya wins the game and if not Hilary wins the game.

(a) Explain why, if Freya chooses $a = 2017$, Hilary cannot win the game. (1 mark)

(b) If Freya chooses $a = 2015$, for how many values of b will Hilary win the game? (2 marks)

(c) For how many values of a is it guaranteed that Freya will win the game, no matter Hilary's choice of b? (7 marks)

You should make it clear which values of a are included in your count, why Freya always wins for those values of a, and how Hilary can win for all other values of a.

Solution

(See the official solutions document.)

Markers' comments

A lot of candidates engaged really well with this question, with around a sixth scoring full marks and another sixth scoring 8 or 9. The mean and median mark were both 5. On the other hand, over a quarter scored no marks. Of the rest, most realised what was going on (that they needed to look at the multiples of the last digit of Freya's number), but were not able to explain this clearly.

There were some clear and succinct write-ups, including several excellent uses of more advanced modular arithmetic results. What was most pleasing, however, was to see so many candidates using basic arithmetic and simple language of last digits to produce detailed and careful explanations.

Many candidates who probably thought they had solved the problem received between 6 and 8 marks. Some of these were due to small oversights, such as miscounting the numbers (in both parts (b) and (c)), or forgetting to count them at all, or confusing the two players (this was not heavily penalised).

The most costly mistake, however, was forgetting that there were two statements to be proved:

(1) When a ends in $1, 3, 7$ or 9, Freya wins for all values of b.

(2) When a ends in $0, 2, 4, 5, 6$ or 8, there are values of b which allow Hilary to win.

Out of the candidates who only proved one of these statements, a majority proved only (1), although there was a considerable number of those who proved only (2). It is really important to realise that these two statements are not equivalent, and thus both need to be explicitly proved.

In fact, many candidates weren't at all clear which of the two statements they were proving. It was common to see explanations such as: "For Freya to guarantee a win, the last digit of a must be $1, 3, 7$ or 9. This is because the multiples of those numbers can end in any digit." The first sentence says the same thing as Statement (2) (if a does not end in those digits, then Freya can lose), but the second sentence is a proof of Statement (1).

Similarly, it is not sufficient to say "Freya can win when a ends in $1, 3, 7, 9$ because their multiples can end in any digit, so one of them will add with the last digit of b to make 10. Multiples of $0, 2, 4, 5, 6, 8$ don't end in all digits, so Freya is not guaranteed to win." This implicitly assumes that the *only way* Freya can win is if multiples of a end in all digits (Statement (2)), but the explanation in the first sentence does not prove that; it only proves that this is *one of the ways* Freya can win (Statement (1)). It is actually quite challenging to write down a simple argument which proves both statements (1) and (2) at the same time. The easiest way to prove Statement (2) is to simply give one example, such as $b = 1$, for which Hilary can win in those cases.

Scripts which only proved one of the two statements were likely to score 7 or 8 marks, so candidates were still rewarded for understanding the essence of the problem. However, there will be olympiad questions where such an omission would be more costly. Logical precision is the cornerstone of mathematics and we would strongly encourage students to think about these issues carefully.

We observed some misconceptions to do with digits and multiples, that may be of interest to teachers. One was to confuse the numbers ending in, for example, 3 with multiples of 3. Another similar one is to think that all numbers ending in a prime digit are prime. (And it would be interesting to ask students whether the converse of this is true!)

There was some confusion over whether 0 counts as a digit, as suggested by the commonly seen statement that "multiples of 7 can end in all digits from 1 to 9", and a clear avoidance of the case of b ending in 0.

These may appear to be small details in the context of school mathematics, but in more advanced work, and not just in mathematics, such "edge cases" become increasingly important and thinking about them would be of great benefit to students.

Question 5

This question requires full written explanations.

(a) Given that m is a positive integer,

(i) What are the possibilities for the last digit of 2^m?

(ii) What are the possible remainders when 2^m is divided by 3?

(2 marks)

(b) Find all positive integer values of n, a and b, with $a \leq b$, such that $n! = 2^a + 2^b$. Justify carefully why there are no other possibilities.

(8 marks)

Note: For a positive integer n we define $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

Solution

(See the official solutions document.)

Markers' comments

This was a difficult problem, which required careful combining of several results to produce a full solution, and we were genuinely impressed that almost 10% of all candidates made serious progress in part (b), with 120 achieving full marks.

Many followed the hints to look at divisibility by 3 and 10 (or 5) and made insightful observation about the last digits of 2^a and 2^b. Some where then able to use the pattern of the last digit to deduce that a and b must differ by an even number, but were not able to combine this with the pattern of remainders when divided by 3, which would quickly lead to the conclusion that n can't be greater than 4.

We saw many very well written solutions, most using the hints from part (a), but some using divisibility by 7 instead. It was also great to see a large number of students engage with the question by answering part (a) and investigating part (b) enough to find the two solutions. Some saw the form of the two valid solutions and just assumed that a and b would have to be consecutive numbers, but that can't be assumed. However, trying different numbers and spotting patterns is a useful starting strategy, even if it does not count as proof.

The most common mistake in this question was including 0 as a possible value of a or b; students should be reminded that zero does not count as a positive number. Another common mistake was to say that remainders on division by 3 are $\frac{1}{3}$ and $\frac{2}{3}$; remainder is defined to be a whole number.

Some students thought that numbers getting bigger and gaps growing meant that there would be no further solutions – but that's not a proof.

Some candidates didn't think carefully enough about whether 0 can actually occur as the last digit of a power of 2, or as a remainder when a power of 2 is divided by 3. Some argued that, because 2^m is even, the last digit can only be $0, 2, 4, 6$ or 8; this is of course, true, but it does not follow that all those digits actually occur.