Topic Test: OxfordAQA A Level Mathematics

Statistics

Name: ____________________

Class: ____________________

Date: ____________________

Time: **92 minutes**

Marks: **76 marks**

Comments:

1 (a) The number of telephone calls, X, received per hour for Dr Able may be modelled by a Poisson distribution with mean 6.

Determine $P(X = 8)$.

(2)

(b) The number of telephone calls, Y, received per hour for Dr Bracken may be modelled by a Poisson distribution with mean λ and standard deviation 3.

(i) Write down the value of λ.

(1)

(ii) Determine $P(Y > \lambda)$.

(2)

(c) (i) Assuming that X and Y are independent Poisson variables, write down the distribution of the **total** number of telephone calls received per hour for Dr Able and Dr Bracken.

(1)

(ii) Determine the probability that a total of at most 20 telephone calls will be received during any one-hour period.

(1)

(iii) The total number of telephone calls received during each of 6 one-hour periods is to be recorded. Calculate the probability that a total of at least 21 telephone calls will be received during exactly 4 of these one-hour periods.

(3)

(Total 10 marks)

2 The continuous random variable X has a cumulative distribution function defined by

$$F(x) = \begin{cases} 0 & x < -5 \\ \dfrac{x+5}{20} & -5 \leq x \leq 15 \\ 1 & x > 15 \end{cases}$$

(a) Show that, for $-5 \leq x \leq 15$, the probability density function, $f(x)$, of X is given by $f(x) = \dfrac{1}{20}$.

(1)

(b) Find:

(i) $P(X \geq 7)$;

(1)

(ii) $P(X \neq 7)$;

(1)

(iii) $E(X)$;

(1)

(iv) $E(3X^2)$.

(3)
(Total 7 marks)

3 A random variable X has an exponential distribution with probability density function $f(x)$, where

$$f(x) = \begin{cases} ke^{-kx} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and k is a constant.

(a) Given that $E(X) = \frac{1}{k}$, find:

(i) using integration, $E(X^2)$;

(ii) $\text{Var}(X)$.

(6)

(b) (i) Derive the cumulative distribution function, $F(x)$, of X for $x \geq 0$.

(3)

(ii) Hence find, in terms of k, the **exact** value of the 90th percentile of X.

(3)

(c) A machine has two essential components, the lifetimes of which follow exponential distributions with means a hours and $3a$ hours. The machine will stop if either component fails. The failures of the two components may be taken to be independent.

Find the probability that the machine continues to work for at least a hours from the start, giving your answer in the form e^q, where q is a rational number to be determined.

(4)
(Total 16 marks)

4 In large-scale tree-felling operations, a machine cuts down trees, strips off the branches and then cuts the trunks into logs of length X metres for transporting to a sawmill.

It may be assumed that values of X are normally distributed with mean μ and standard deviation 0.16, where μ can be set to a specific value.

(a) Given that μ is set to 3.3, determine:

(i) $P(X < 3.5)$;

(3)

(ii) $P(X > 3.0)$;

(3)

(iii) $P(3.0 < X < 3.5)$.

(2)

(b) The sawmill now requires a batch of logs such that there is a probability of 0.025 that any given log will have a length less than 3.1 metres.

Determine, to two decimal places, the new value of μ.

(4)

(Total 12 marks)

5 Becky owns a taxi. Each weekday morning, she collects Steve from his home and takes him to the train station.

A record of the times, x minutes, for a random sample of 65 such taxi journeys is summarised by

$$\sum x = 1326.0 \text{ and } \sum (x - \bar{x})^2 = 400.24$$

(a) (i) Calculate the value of the sample mean, x.

(1)

(ii) Show that, correct to two decimal places, $s = 2{:}50$, where s^2 denotes the unbiased estimate of the population variance.

(2)

(b) (i) Construct a 96% confidence interval for the mean journey time.

(4)

(ii) State why use of the Central Limit Theorem was required in calculating this confidence interval.

(1)

(c) Comment on Becky’s claim that the mean journey time is more than 20 minutes.

(2)

(Total 10 marks)

6 The times taken to complete a round of golf at Slowpace Golf Club may be modelled by a random variable with mean μ hours and standard deviation 1.1 hours.

Julian claims that, on average, the time taken to complete a round of golf at Slowpace Golf Club is greater than 4 hours.

The times of 40 randomly selected completed rounds of golf at Slowpace Golf Club result in a mean of 4.2 hours.

(a) Investigate Julian's claim at the 5% level of significance.

(6)

(b) If the actual mean time taken to complete a round of golf at Slowpace Golf Club is 4.5 hours, determine whether a Type I error, a Type II error or neither was made in the test conducted in part (a). Give a reason for your answer.

(2)

(Total 8 marks)

7 A supermarket buys pears from a local supplier. The supermarket requires the mean weight of the pears to be at least 175 grams. William, the fresh-produce manager at the supermarket, suspects that the latest batch of pears delivered does not meet this requirement.

(a) William weighs a random sample of 6 pears, obtaining the following weights, in grams.

160.6 155.4 181.3 176.2 162.3 172.8

Previous batches of pears have had weights that could be modelled by a normal distribution with standard deviation 9.4 grams. Assuming that this still applies, show that a hypothesis test at the 5% level of significance supports William's suspicion.

(7)

(b) William then weighs a random sample of 20 pears. The mean of this sample is 169.4 grams and $s = 11.2$ grams, where s^2 is an unbiased estimate of the population variance.

Assuming that the population from which this sample is taken has a normal distribution but with unknown standard deviation, test William's suspicion at the **1%** level of significance.

(5)

(c) Give a reason why the probability of a Type I error occurring was smaller when conducting the test in part (b) than when conducting the test in part (a).

(1)

(Total 13 marks)

Mark schemes

1 (a) $P(X = 8) = P(X \le 8) - P(X \le 7)$
$= 0.8472 - 0.7440$

$$P(X = 8) = \frac{e^{-6}(6^8)}{8!}$$

M1

$= 0.103$

A1

2

(b) (i) $\lambda = 9$

B1

1

(ii) $P(Y > 9) = 1 - P(Y \le 9)$
$= 1 - 0.5874$

M1

$= 0.4126$

AWFW 0.412 to 0.413

A1ft

2

(c) (i) $T \sim \text{Po}(15)$

B1ft

1

(ii) $P(T \le 20) = 0.917$

B1ft

1

(iii) P(T at least 21) = 0.083

B1ft

p = 15 × $(0.083)^4(0.917)^2$

B(6, (iii)) used

M1

= 0.000599

CAO; AWFW 0.000598 to 0.0006

A1

3

[10]

2 (a) for $-5 \le x \le 15$

$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x+5}{20}\right) = \frac{1}{20}$

AG

B1

1

(b) (i) P($X \ge 7$) = 1 − F(7) = 1 − $\frac{12}{20} = \frac{2}{5}$ **or** $\left[\frac{8}{20}; \frac{4}{10}; 0.4\right]$

Alternative:

Use of $f(x) = \frac{1}{20}$ ***or*** *graph* ⇒

P($X \ge 7$) = $\frac{1}{20}$ × *(15 − 7)* = $\frac{2}{5}$ *(oe)*

B1

1

(ii) P($X \ne 7$) = 1

cao

B1

1

(iii) E(X) = $\frac{1}{2}(-5 + 15) = 5$

Alternative:

$E(X) = \int_{-5}^{15} \frac{x}{20} \mathrm{d}x = \left[\frac{x^2}{40}\right]_{-5}^{15} = \frac{1}{40}(225 - 25) = \frac{1}{40} \times 200; = 5$

***B1** (cao)*

B1

1

(iv) E($3X^2$) = $\left.\int_{-5}^{15} \frac{3x^2}{20} \mathrm{d}x\right\}$ (ignore limits)

M1

$$\left.\begin{array}{l}\left[\frac{x^3}{20}\right]_{-5}^{15} \\ \frac{1}{20}(3375 + 125) \\ 168\frac{3}{4} + 6\frac{1}{4}\end{array}\right\}$$

correct limits seen / used

A1

= 175

(cao) (allow 174.9)

A1

3

Alternative:

Var(X) = $\frac{1}{12}(15 - -5)^2 = \frac{400}{12}$ (oe)

$E(3X^2) = 3\,E(X^2)$

(B1)

$E(3X^2) = 3\times\left[\frac{400}{12}+5^2\right]$

$= 3\times\left[\{\text{their Var}(X)>0\}+\{\text{their E}(X)\}^2\right]$ ***used*** *(⇒ M1)*

(M1)

= 175

(A1)

(3)

[7]

3 (a) $E(X^2) = \int_0^\infty kx^2 e^{-kx}\,dx$

M1

$= \left[-x^2e^{-kx}\right]_0^\infty + \int_0^\infty 2xe^{-kx}dx$

M1

$= 0+\left[-\frac{2x}{k}e^{-kx}\right]_0^\infty + \int_0^\infty \frac{2}{k}e^{-kx}dx$

0 may be omitted, or limits inserted at end of process.

(E(X) integral can be quoted.)

A1

$= 0 + \left[-\frac{2}{k^2}e^{-kx}\right]_0^\infty$

Ditto.

A1

$= \frac{2}{k^2}$

A1

$\text{Var}(X) = \frac{2}{k^2}-\left(\frac{1}{k}\right)^2 = \frac{1}{k^2}$

Their E(X^2) minus mean^2, provided ***positive****.*

SC *Allow B1 for those who write correct working and result, having failed to integrate correctly*

A1 √

6

(b) (i) F (x) = $\int_0^x k e^{-ku} du$

M1

$\left[-e^{-ku}\right]_0^x$ = 1 - e^{-kx}

A1A1

3

(ii) $\left[1 - e^{-kx}\right]_0^N$ = 0.9 ⇒ $e^{-kN} = 0.1$

M1

⇒ $N = \frac{1}{k} \ln 10$

M1 for taking logs. cao, acf

M1A1

3

(c) Mean = $a = \frac{1}{k}$ ⇒ $k = \frac{1}{a}$,

M1A1

Mean = $3a = \frac{1}{k} \Rightarrow k = \frac{1}{3a}$

$e^{-\frac{1}{a}a} \cdot e^{-\frac{1}{3a} \cdot a} = e^{-1} \cdot e^{-\frac{1}{3}} = e^{-\frac{4}{3}}$

cwo

M1A1

4

[16]

4

(a) (i) $P(X < 3.5) = P\left(Z < \frac{3.5 - 3.3}{0.16}\right) =$

Standardising (3.45, 3.5 or 3.55) with 3.3

& ($\sqrt{0.16}$, 0.16 or 0.16^2) and/or (3.3 – x)

M1

$P(Z < 1.25) =$

CAO; ignore sign

A1

0.894 to 0.895

AWFW (0.89435)

A1

3

(ii) $P(X > 3.0) = P\left(Z > \frac{3.0 - 3.3}{0.16}\right) =$

Standardising (2.95, 3 or 3.05) with 3.3

& ($\sqrt{0.16}$, 0.16 or 0.16^2) and/or (3.3 – x)

M1

$P(Z > -1.875) = P(Z < 1.875) =$

Correct area change

m1

0.969 to 0.97(0)

AWFW (0.96960)

A1

3

(iii) $P(3.0 < X < 3.5) =$ (i) – [1 – (ii)] =

OE

M1

0.863 to 0.865

AWFW: CSO (0.86395)

A1

2

(b) 0.025 $\Rightarrow z = 1.96$

CAO; ignore sign

B1

$z = \frac{3.1 - \mu}{0.16}$

Standardising 3.1 with μ and 0.16;
allow (μ – 3.1)

M1

= –1.96

Equating z-term to z-value;
not using 0.025, 0.975, |1 – z|
or Φ(0.025) = 0.507 to 0.512

m1

Hence μ = 3.4(0) to 3.42

AWFW; CSO (3.4136)

A1

4

[12]

5 (a) (i) Mean, $\bar{x} = 20.4$

CAO

B1

1

(ii) Standard deviation $s = \sqrt{\frac{400.24}{64}}$

Expression must be seen $\left(\sqrt{6.25375}\right)$

M1

= 2.50

AWRT (2.50075)

A1

2

NB: $s = \frac{400.24}{64 \text{ or } 65}$ (6.15754 or 6.25375)

No $\sqrt{\ }$ *and / or use of divisor n*

(B1)

or

$s = \frac{\sqrt{400.24}}{65}$ (2.48144)

Use of divisor n

(B1)

(b) (i) 96%(0.96) $\Rightarrow z$ = 2.05 to 2.06

AWFW (2.0537)

B1

CI for μ is $\bar{x} \pm z \times \frac{s}{\sqrt{n}}$

Used

Must have $\sqrt{n}$ *with n > 1*

M1

Thus $20 \pm 2.0537 \times \frac{2.50}{\sqrt{65 \text{ or } 64}}$

F on $\bar{x}$ and z

A1F

Hence 20.4 ± 0.6 **or** (19.8, 21(.0))
AWRT

A1

4

(ii) Times / *X* are not (known to be) normally distributed
Or equivalent
Not data, values, sample, n large

B1

1

(c) CI in (b)(i) contains/includes 20
Or equivalent
Dependent on CI in (b)(i)

B1F

thus

no (significant) evidence to support claim
Or equivalent
Dependent on B1F

Bdep1

2

[10]

6 (a) H_0: $\mu = 4.0$

H_1: $\mu > 4.0$
(both)

B1

$z_{calc} = \frac{4.2 - 4}{1.1/\sqrt{40}}$

Alternative:
P($\bar{X}$ > 4.2) = P(Z > 1.15) ***M1A1***

M1

= 1.15

awrt

A1

z_{crit} = 1.6449

= 1 – 0.87493

= 0.125 ***B1***

0.125 > 0.05 ⇒ accept H_0 ***Adep1***

B1

Accept H_0 [or Reject H_1]

Dep on B1M1B1

A1

Insufficient evidence at 5% level to support Julian’s claim

Dep on previous mark

E1

6

(b) Type II error.

Follow through on conclusion in (a)

B1ft

Accepted H_0 when H_0 was false (oe)

Dep on previous mark

E1

If Reject H_0 in (a) then:

No error (B1ft)

Rejected H_0 when H_0 was false (oe) (E1)

2

[8]

7 (a) $H_0: \mu = 175$

$H_1: \mu < 175$

Both; accept H0: $\mu \geq 175$

Do not accept mean or $\overline{x}$

but accept population mean

B1

$\overline{x} = 168.1$

B1

$$z = \frac{\text{‘}168.1\text{’} - 175}{9.4/\sqrt{6}}$$

For use of 9.4 / √6

M1

For rest of formula (ignore sign)

m1

$= -1.798$

Must be negative AWRT −1.80

A1

CV = − 1.6449

AWFW − 1.64 to − 1.65

B1

−1.6449 > −1.798 so test statistic in critical region

Comparison of correct test statistic with correct CV ***must be seen*** *(diagram or words)*

Reject H_0, significant evidence that batch **mean** is less than 175 grams

OE; suspicion supported

Must be in context AG

A1

7

(b) H_0: μ = 175

H_1: μ < 175

Award B1 for both correct if not scored in ***(a)***

$$t = \frac{169.4 - 175}{11.2/\sqrt{20}}$$

For use of 11.2 / √20

M1

For rest of formula (ignore sign)

m1

= − 2.236

Must be negative AWRT − 2.24

A1

CV(t_{19}) = − 2.539

AWRT − 2.54

B1

− 2.236 > − 2.539 so test statistic not in critical region

Comparison of correct test statistic with correct CV (need not be seen)

Accept H_0, no significant evidence that batch mean / weight is less than 175 grams

OE; suspicion not supported

A1

5

(c) Because the significance level is 1% instead of 5%

OE; eg SL is different

Reference to sample size ⇒ E0

E1

1

[13]