Topic Test: OxfordAQA International AS Further Mathematics

Pure Mathematics FPSM1

Name: ____________________

Class: ____________________

Date: ____________________

Time: **64 minutes**

Marks: **54 marks**

Comments:

1

(a) The matrix **X** is defined by $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

(i) Given that $\mathbf{X}^2 = \begin{bmatrix} m & 2 \\ 3 & 6 \end{bmatrix}$, find the value of m.

(1)

(ii) Show that $\mathbf{X}^3 - 7\mathbf{X} = n\mathbf{I}$, where n is an integer and $\mathbf{I}$ is the 2×2 identity matrix.

(4)

(b) It is given that $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(i) Describe the geometrical transformation represented by **A**.

(1)

(ii) The matrix **B** represents an anticlockwise rotation through 45° about the origin.

Show that $\mathbf{B} = k\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, where k is a surd.

(2)

(iii) Find the image of the point $P(-1, 2)$ under an anticlockwise rotation through 45° about the origin, followed by the transformation represented by **A**.

(4)

(Total 12 marks)

2

The plane transformation T is defined by

$$\text{T} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(a) A shape has an area of 3 square units. Find the area of the shape after being transformed by T.

(2)

(b) (i) Find the equations of all the invariant lines of T.

(5)

(ii) State the equation of the line of invariant points of T.

(1)

(Total 8 marks)

3

Let $\mathbf{X} = \begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}$.

(a) Determine $\mathbf{X}\,\mathbf{X}^{\mathrm{T}}$.

(2)

(b) Show that Det($\mathbf{X}\,\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X}$) $\leq$ 0 for all real values of x.

(4)

(c) Find the value of x for which the matrix ($\mathbf{X}\,\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X}$) is singular.

(1)

(Total 7 marks)

4 The variables y and x are related by an equation of the form

$$y = ax^n$$

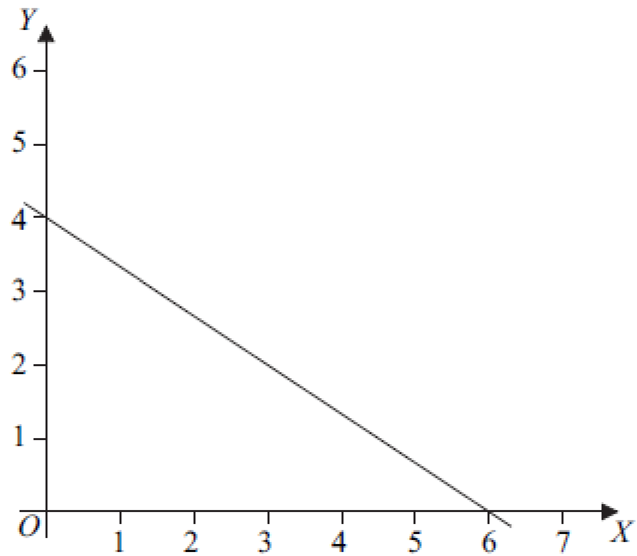
where a and n are constants.

Let $Y = \log_{10} y$ and $X = \log_{10} x$.

(a) Show that there is a linear relationship between Y and X.

(3)

(b) The graph of Y against X is shown in the diagram.



Find the value of n and the value of a.

(4)

(Total 7 marks)

5 The equation

$$24x^3 + 36x^2 + 18x - 5 = 0$$

has one real root, α.

(a) Show that α lies in the interval $0.1 < x < 0.2$.

(2)

(b) Starting from the interval $0.1 < x < 0.2$, use interval bisection **twice** to obtain an interval of width 0.025 within which α must lie.

(3)

(c) Taking $x_1 = 0.2$ as a first approximation to α , use the Newton-Raphson method to find a second approximation, x_2 , to α. Give your answer to four decimal places.

(4)
(Total 9 marks)

6 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \frac{y - x}{y^2 + x}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$.

(3)

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\mathrm{f}(x_r, y_r)$$

with your answer to part (a), to obtain an approximation to $y(1.2)$, giving your answer to three decimal places.

(3)
(Total 6 marks)

7 A curve passes through the point (1, 3) and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{1 + x^3}$$

Starting at the point (1, 3), use a step-by-step method with a step length of 0.1 to estimate the value of y at $x = 1.2$. Give your answer to four decimal places.

(Total 5 marks)

Mark schemes

1 (a) (i) $\mathbf{X}^2 = \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix}$; (*m* =)7

(m =)7 or 7 as top left element of $\boldsymbol{X}^2$

B1

1

(ii) $\mathbf{X}^3 = \begin{bmatrix} 13 & 14 \\ 21 & 6 \end{bmatrix}$;

At least 2 elements correct

M1

$7\mathbf{X} = \begin{bmatrix} 7 & 14 \\ 21 & 0 \end{bmatrix}$

PI

B1

$\mathbf{X}^3 - 7\mathbf{X} = \begin{bmatrix} 13-7 & 14-14 \\ 21-21 & 6-0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

Ft on c's m value

A1F

$= 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 6\mathbf{I}$

CSO Accept either form but at least one must be shown explicitly

A1

4

(b) (i) Reflection in the *x*-axis

OE

B1

1

(ii) $\mathbf{B} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Either OE. For M mark, accept dec. equiv.

(at least 3sf) for $\frac{1}{\sqrt{2}}$

M1

$= \dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

NMS SC1 for $k = \dfrac{1}{\sqrt{2}}$ *or better.*

A1

2

(iii) **AB** $\begin{bmatrix}-1\\ 2\end{bmatrix} = k\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}$

Attempt to find ***AB*** $\begin{bmatrix}-1\\ 2\end{bmatrix}$

M1

$= k\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-3\\ 1\end{bmatrix} \quad \left\{\text{or } k\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}\right\}$

Either $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}-3\\ 1\end{bmatrix}$ *or* $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}$

A1

$= k\begin{bmatrix}-3\\ -1\end{bmatrix}$

Completing the matrix mult. to reach a 2 × 1 matrix

m1

(Image of P is the point) $\left(-\dfrac{3}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

CSO

SC *Wrong order, works with* ***BA*** $\begin{bmatrix}-1\\ 2\end{bmatrix}$,

mark out of a max of M1A0 m1A0

A1

4

[12]

2 (a) Determinant of matrix = −8 + 9 = 1

Finding determinant and multiplying by area

M1

Area = 3 × 1 = 3 (square units)

CAO − must show multiplication or refer to scale factor / invariant area or equivalent

A1

2

(b) (i) $\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x \\ mx+c \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

⇒

$(x') = 4x + 3(mx + c)$

$(y') = -3x - 2(mx + c)$

x', y' in terms of x, y, m, c

M1

Invariant lines ⇒ $y' = mx' + c$

⇒ $-3x - 2mx - 2c = 4mx + 3m^2 x + 3mc + c$

⇒ $0 = (3m^2 + 6m + 3)x + 3mc + 3c$

Use of $y' = mx' + c$

A1

⇒ $3m^2 + 6m + 3 = 0 \quad 3mc + 3c = 0$

Attempt at solving equations where coefficients = 0 or compares coefficients

M1

$3(m + 1)^2 = 0 \qquad 3c(m + 1) = 0$

⇒ $m = -1$ $\quad c$ can be any value

Finding the correct value of m

A1

⇒ lines are $y = -x + c$

Fully correct line − no restriction on c

A1

5

SPECIAL CASES

$$\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ -x+c \end{pmatrix}=\begin{pmatrix} x+3c \\ -x-2c \end{pmatrix}$$

$x' = x + 3c$

$y' = -x - 2c$

SC1 − Correct multiplication as shown

Consider

$-x' + c$
$= -(x + 3c) + c$
$= -x - 3c + c$
$= -x - 2c$
$= y'$

Hence $y = -x + c$ is an invariant line

SC2 − correct multiplication as shown above and full algebraic solution using $y' = -x' + c$

(ii) When $c = 0$, $y = -x$ is a line of invariant points

Any equivalent form

B1

1

[8]

3 (a) $\mathbf{X}\mathbf{X}^{\mathrm{T}} = \begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ x & 7 \end{bmatrix}$

Attempted multn. with $\mathbf{X}^{T}$ correct

M1

$= \begin{bmatrix} x^2+9 & 7x-3 \\ 7x-3 & 50 \end{bmatrix}$

A1

2

(b) $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \begin{bmatrix} 3 & -1 \\ x & 7 \end{bmatrix}\begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 3x-7 \\ 3x-7 & x^2+49 \end{bmatrix}$

Good attempt

M1

$\mathbf{X}\mathbf{X}^T - \mathbf{X}^T\mathbf{X} = \begin{bmatrix} x^2-1 & 4x+4 \\ 4x+4 & 1-x^2 \end{bmatrix}$

Good attempt

M1

$\text{Det}(\mathbf{X}\mathbf{X}^T - \mathbf{X}^T\mathbf{X}) = \begin{bmatrix} x^2-1 & 4x+4 \\ 4x+4 & 1-x^2 \end{bmatrix} = (x+1)^2 \begin{vmatrix} x-1 & 4 \\ 4 & 1-x \end{vmatrix}$

Good attempt to factorise / expand the determinant

M1

$= -(x+1)^2\{(x-1)^2 + 16\} \leq 0$
for all real x

Explained / demonstrated fully

E1

4

(c) $x = -1$

CSO

B1

1

[7]

4 (a) $y = ax^n \Rightarrow \log_{10} y = \log_{10} ax^n$

$\log_{10} y = \log_{10} a + \log_{10} x^n$

Take logs and apply one log law in soln. correctly PI.

M1

$\log_{10} y = \log_{10} a + n\log_{10} x$

Apply a further log law correctly.

m1

$Y = \log_{10} a + nX$ (which is a linear relationship between Y and X.)

Correct eqn. with base 10 (or lg or later evidence of use of base 10 if log without base here)

A1

3

(b) n = gradient of line

Stated or used.

Accept $n = \pm\frac{2}{3}$ OE as evidence

M1

$n = -\frac{2}{3}$

$n = -\frac{2}{3}$(OE 3sf)

A1

$\log_{10} a = 4$

Equating c's constant term [must involve a log] in c's (a) eqn. to the Y-intercept value PI by correct value of a

M1

$a = 10^4$ (=10 000)

A1

4

[7]

5 (a) Let $f(x) = 24x^3 + 36x^2 + 18x - 5$

f(0.1) = −2.816, f(0.2) = 0.232

Both attempted and at least one evaluated correctly to at least 1sf rounded or truncated OE fraction

M1

Change of sign so α lies between 0.1 and 0.2

Need both evaluations correct to above degree of accuracy and 'change of sign OE' <u>*and*</u> *relevant reference to 0.1 and 0.2*

A1

2

(b) f(0.15) = −1.409 (< 0 so root > 0.15)

f(0.15) considered first

M1

f(0.175) ≈ −0.619 (< 0 so root > 0.175)

f(0.15) then f(0.175) both evaluated correctly to at least 1sf OE fractions

A1

α lies between 0.175 and 0.2

Dependent on both previous marks gained and no other additional evaluations other than at 0.15 and 0.175

A1

3

(c) $f'(x) = 72x^2 + 72x + 18$

PI

B1

$(x_2 =)$

$$0.2 - \frac{24(0.2)^3 + 36(0.2)^2 + 18(0.2) - 5}{72(0.2)^2 + 72(0.2) + 18}$$

B1 for numerator in correct formula

B1

B1 for denominator in correct formula

B1

= 0.1934 (to 4dp)

CAO Must be 0.1934

Do not apply ISW

NMS scores 0 / 4

B1

4

[9]

6

(a) $y(1.1) = y(1) + 0.1\left[\frac{2-1}{4+1}\right]$

M1A1

= 2 + 0.02 = 2.02

A1

3

(b) $y(1.2) = y(1) + 2(0.1)\{f[1.1, y(1.1)]\}$

M1

$= 2 + 2(0.1)\left[\frac{2.02 - 1.1}{2.02^2 + 1.1}\right]$

ft on c's answer to (a)

A1F

= 2.035518… = 2.036 to 3dp

CAO Must be 2.036

A1

3

[6]

7 $y_{n+1} \approx y_n + h\,\mathrm{f}(x_n)$

OE

$h\,y'(1) = 0.1 \times y'(1)$ (=0.05)

Attempt to find $h\,y'(1)$. PI by eg 3.05 for y(1.1)

M1

$y(1.1) \approx 3 + 0.05 = 3.05$

A1

$y(1.2) \approx y(1.1) + 0.1 \times y'(1.1) = 3.05 + 0.1 \times y'(1.1)$

$\approx 3.05 + 0.1 \times \dfrac{1.1}{1+1.1^3} \quad \left(= 3.05 + 0.1 \times \dfrac{1100}{2331}\right)$

Attempt to find y(1 + 0.1) + 0.1 × y′(1 + 0.1) must see evidence of calculation if correct ft [0.047.. + c's y(1.1)] value not obtained

m1

$\approx 3.05 + 0.047(19.....)$

OE; ft on [0.047..+ c's y(1.1)] value; PI

A1F

$\approx$ 3.0972 (to 4 d.p.)

Must be 4 dp.

A1

[5]