1. (a) Express $10\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$
Give the exact value of R and give the value of α, in degrees, to 2 decimal places.

(3)

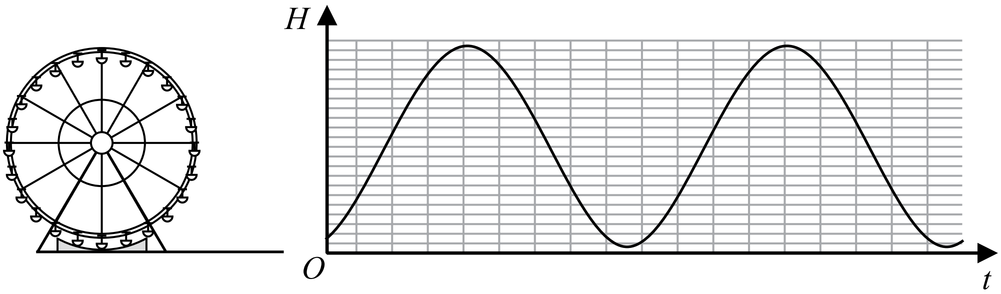


Figure 3

The height above the ground, H metres, of a passenger on a Ferris wheel t minutes after the wheel starts turning, is modelled by the equation

$$H = a - 10\cos(80t)° + 3\sin(80t)°$$

where a is a constant.

Figure 3 shows the graph of H against t for two complete cycles of the wheel.

Given that the initial height of the passenger above the ground is 1 metre,

(b) (i) find a complete equation for the model,

(ii) hence find the maximum height of the passenger above the ground.

(2)

(c) Find the time taken, to the nearest second, for the passenger to reach the maximum height on the second cycle.

(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(3)

It is decided that, to increase profits, the speed of the wheel is to be increased.

(d) How would you adapt the equation of the model to reflect this increase in speed?

(1)

a) $R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$10\cos\theta - 3\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$10\cos\theta = R\cos\theta\cos\alpha$ $\quad\quad$ $-3\sin\theta = -R\sin\theta\sin\alpha$

$R\cos\alpha = 10$ $-$ 1 $\quad\quad$ $R\sin\alpha = 3$ $-$ 2

$R^2\cos^2\alpha = 100$ $\quad\quad$ $R^2\sin^2\alpha = 9$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 100 + 9$ $\quad\quad$ $\cos^2\alpha + \sin^2\alpha \equiv 1$

$R^2(\cos^2\alpha + \sin^2\alpha) = 109$

$R^2 = 109$

a) $R = \sqrt{109}$ – 1

② ÷ ①

$$\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \frac{3}{10}$$

$$\tan\alpha = \frac{3}{10} \quad - 1$$

$$\alpha = 16.70° \text{ (2d.p.)} \quad - 1$$

$$= \sqrt{109}\cos(\theta + 16.70°)$$

b) i) $1 = a - 10\cos(0) + 3\sin(0)$

$1 = a - 10$

$a = 11$

$$H = 11 - 10\cos(80t)° + 3\sin(80t)°$$

$10\cos\theta - 3\sin\theta$

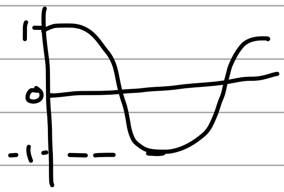
$= \sqrt{109}\cos(\theta + 16.70°)$

$-10\cos(80t)° + 3\sin(80t)°$

$= -(10\cos(80t)° - 3\sin(80t)°)$

$= -\sqrt{109}\cos(80t + 16.70)°$

$$H = 11 - \sqrt{109}\cos(80t + 16.70)° \quad - 1$$



ii) $\cos(80t + 16.70)° = -1$

$H = 11 - (\sqrt{109} \times -1)$

$H = 11 + \sqrt{109}$ – 1

c) $\cos(80t + 16.70)° = -1$

$80t + 16.70 = 180$ for first cycle

$180 + 360 = 540$ for second cycle

$80t + 16.70 = 540$ – 1

$$t = \frac{540 - 16.7}{80} \quad - 1$$

0.54×60

$= 32$ (2s.f)

c) $t = 6.54$ minutes

$t = 6$ minutes 32 seconds - 1

d) $H = 11 - \sqrt{109}\cos(80t + 16.70)^\circ$

We would increase the '80t' value - for example to 90t. - 1

2. The depth of water, D metres, in a harbour on a particular day is modelled by the formula

$$D = 5 + 2\sin(30t)° \qquad 0 \leqslant t < 24$$

where t is the number of hours after midnight.

A boat enters the harbour at 6:30 am and it takes 2 hours to load its cargo.
The boat requires the depth of water to be at least 3.8 metres before it can leave the harbour.

(a) Find the depth of the water in the harbour when the boat enters the harbour.

(1)

(b) Find, to the nearest minute, the earliest time the boat can leave the harbour.
(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(4)

a) 6:30 am → 6h 30min or 6.5h

$5 + 2\sin(30 \times 6.5)$

$= 4.48\,\text{m}$ ✓

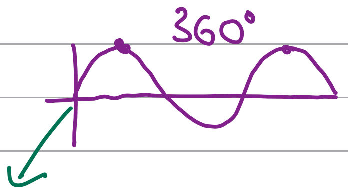
b) $D = 5 + 2\sin(30t)$ find at what t is $D = 3.8\,\text{m}$

$3.8 = 5 + 2\sin(30t)$

$-1.2 = 2\sin(30t)$

$-0.6 = \sin(30t)$

$\sin(30t) = -0.6$ ✓



$30t = \sin^{-1}(-0.6) = -36.869\ldots$ $\quad t > 0$

$30t = -36.86\ldots + 360$ ✓ $\quad \therefore 30t > 0$

$= 323.13\ldots$

$t = \dfrac{323.13\ldots}{30} = 10.77\text{h}$ ✓ $\to 10\text{h}\ 0.77 \times 60\,\text{min}$

$= 10\text{h}\ 46\,\text{min}$

the boat can leave at 10:46 am ✓

$$f(x) = 10e^{-0.25x} \sin x, \qquad x \geqslant 0$$

3. (a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

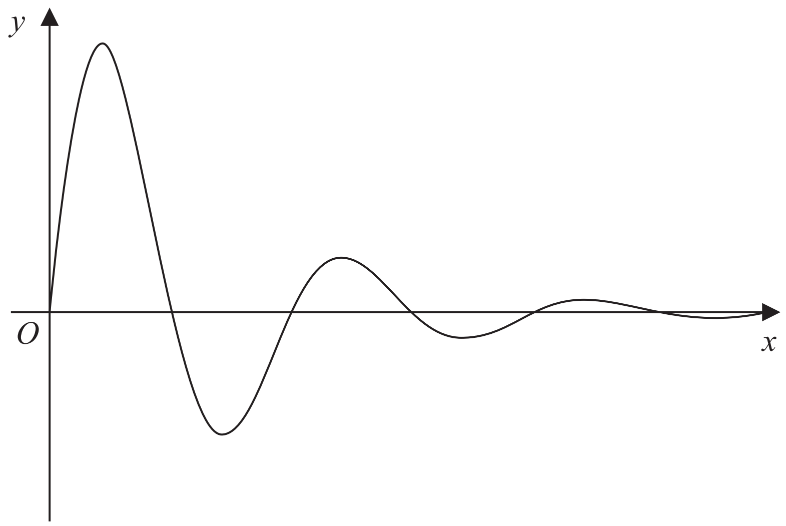


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

(b) Sketch the graph of H against t where

$f(x)$

$$H(t) = \left|10e^{-0.25t} \sin t\right| \qquad t \geqslant 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce.

(3)

(d) Explain why this model should not be used to predict the time of each bounce.

(1)

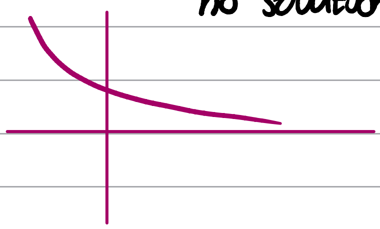
a) $f(x) = 10e^{-0.25x}\sin x$

$f'(x) = -0.25(10e^{-0.25x})\sin x + \cos x(10e^{-0.25x})$
$= -2.5\sin x e^{-0.25x} + 10\cos x e^{-0.25x}$ 2

$0 = -2.5\sin x e^{-0.25x} + 10\cos x e^{-0.25x}$
$= e^{-0.25x}(-2.5\sin x + 10\cos x)$ 1

$\therefore\ e^{-0.25x} = 0$ or $-2.5\sin x + 10\cos x = 0$

reject because no solution



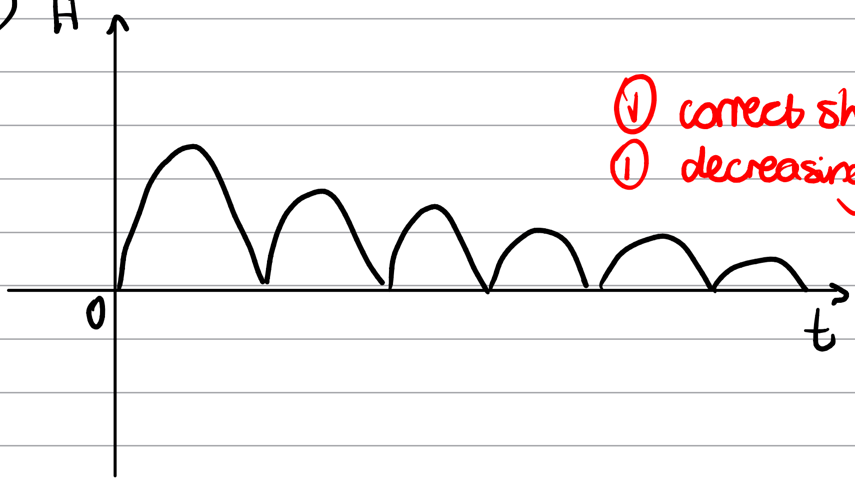
$10\cos x = 2.5\sin x$

$\frac{10}{2.5} = \frac{\sin x}{\cos x}$ 1

$\frac{\sin x}{\cos x} = \tan x$

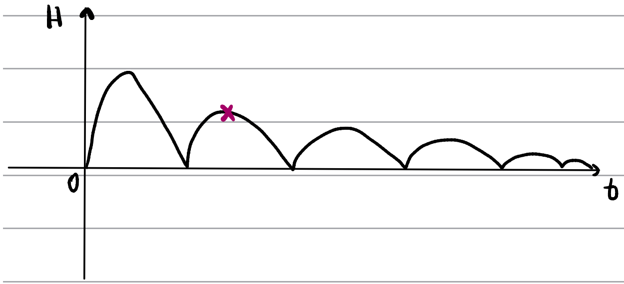
$\tan x = 4$ as needed

b)



1 correct shape
1 decreasing heights

c)



(1)

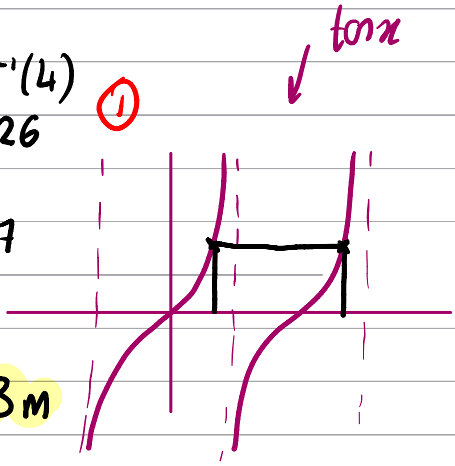
$\tan x = 4$

$x = \tan^{-1}(4)$

$= 1.326$ (1)

$+\pi$

$= 4.47$



$H(4.47) = \left|10e^{-0.25(4.47)} \times \sin(4.47)\right| = 3.18\,m$

(1)

d)

The times between each bounce should not stay the same when the heights of each bounce is getting smaller (1)

4. (a) Express $\sin x + 2\cos x$ in the form $R\sin(x + \alpha)$ where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

a) $\sin x + 2\cos x \longrightarrow R\sin(x+\alpha)$

$\boxed{1}$ Find α

$\boxed{2}$ Find R

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha \Rightarrow \sin x = R\sin x\cos\alpha \Rightarrow R\cos\alpha = 1$

$2\cos x = R\cos x\sin\alpha \Rightarrow R\sin\alpha = 2$

$\Rightarrow \tan\alpha = \frac{2}{1} \Rightarrow \alpha = \tan^{-1}(2)$

$\alpha = 1.10714...$ 1 $\Rightarrow \alpha = \underline{1.107}$ (3 d.p) 1

$R = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$ 1

$\Rightarrow \alpha = \underline{1.107}$ (radians), $R = \underline{\sqrt{5}}$ $\Rightarrow \sin x + 2\cos x = \underline{\sqrt{5}}\sin(x + 1.107)$

The temperature, θ°C, inside a room on a given day is modelled by the equation

$$\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right) \qquad 0 \leqslant t < 24$$

where t is the number of hours after midnight.

Using the equation of the model and your answer to part (a),

(b) deduce the maximum temperature of the room during this day,

(1)

b) $\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right)$

let $x = \frac{\pi t}{12} - 3$, we can use our answer from part a. $\left(\sqrt{5}\sin(x + 1.107) = \sin x + 2\cos x\right)$

$\Rightarrow \theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$, we have a maximum when $\sin x = 1$

$\Rightarrow \theta = (5 + \sqrt{5})^\circ C$ or $\theta = \underline{7.24}\ ^\circ C$ (3 s.f) 1

(c) find the time of day when the maximum temperature occurs, giving your answer to the nearest minute.

(3)

c) $\theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$

In part b, we said the maximum temperature occurs when $\sin x = 1$.

$\Rightarrow x = \sin^{-1}(1)$

$\Rightarrow x = \underline{\pi/2}$

$\Rightarrow \frac{\pi t}{12} - 3 + 1.107 = \frac{\pi}{2}$

$\Rightarrow \frac{\pi t}{12} = \frac{\pi}{2} + 3 - 1.107$

0.2 of an hour is equal $0.2 \times 60 = 12$ mins

$\Rightarrow \cancel{\pi}t = \frac{12\left(\frac{\pi}{2} + 3 - 1.107\right)}{\pi}$ 1 $\Rightarrow t = 13.2$ hours 1

$\Rightarrow t = 13$ hours and 12 minutes after midnight. 1

5. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

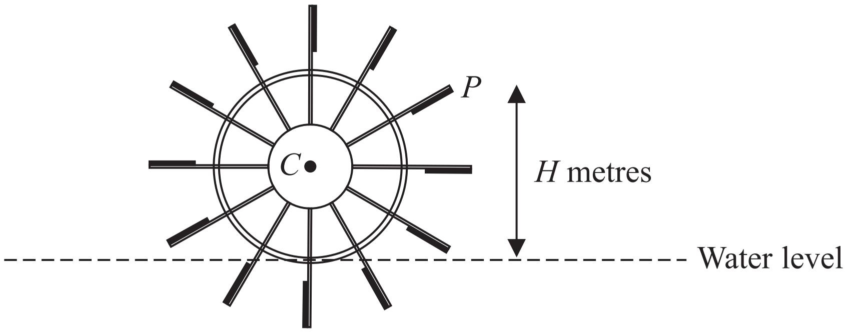


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

Question 5 continued

a) $2\cos\theta - \sin\theta = R\cos(\theta + \alpha)$

$a\cos\theta \pm b\sin\theta$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$ convert using formula:

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$R = \sqrt{a^2 + b^2}$

$= \sqrt{2^2 + (-1)^2}$

$= \sqrt{5}$ 1

to find α, compare like terms:

$\cos\theta$: $2 = R\cos\alpha$

$\cos\alpha = 2/R$

$\sin\theta$: $1 = R\sin\alpha$

$\sin\alpha = 1/R$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{1/R}{2/R}$

$= \dfrac{1}{2}$

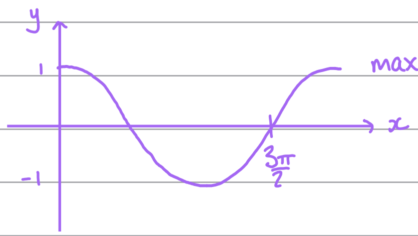
$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$ (3 d.p)

[must be within given range]

b) i) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$

$H = 3 + 2[2\cos(0.5t) - \sin(0.5t)]$ using answer from a)

$H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$



max $\cos x = 1$ so max $H = 3 - 2\sqrt{5}(1)$

$H_{max} = 3 + 2\sqrt{5}$ 1

Question 5 continued

ii) $\cos(0.5t + 0.464) = 1$ ← using max value from graph

$0.5t + 0.464 = \cos^{-1}(1) = 2\pi$ (1)

$0.5t = 2\pi - 0.464$

$t = 2(2\pi - 0.464)$

$t = 11.6\text{ s}$ (1)

c) $3 + 2\sqrt{5}\cos(0.5t + 0.464) = 0$

$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$ (1)

$0.5t + 0.464 = \cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$

$t = 2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right) - 0.464\right)$ (1)

So the time required is. (1)

$2(3.977 - 0.464) - 2(2.306 - 0.464) = 3.34$ (1)

d) The '3' would need to vary. (1)